Mathematics 3.5: Apply the algebra of complex numbers in solving problems

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1 Complex numbers in rectangular form

The path through your High-School mathematics education has been paved, quite literally, with scores of unspeakable lies and simplifications. One of the more grievous of these is the seemingly-legitimate notion that an expression such as $\sqrt{-4}$ *cannot be evaluated. In fact, no. As mathematics progressed, this unfortunate theoretical cul-de-sac caused increasing - and understandable- levels of anxiety in certain circles. Things were eventually, though not necessarily painlessly - resolved, which leads us to our current predicament of preparing for a ghastly external examination.*

Manipulation of surds

We begin these notes with a idea that is, in all likelihood, insultingly elementary to you (bear with us, however). The idea is that there are some integers whose *square root* can be directly evaluated. One example might be $\sqrt{25}$, which, clearly, can be expressed as the number 5. Such numbers are called *perfect squares.* While working with perfect squares is obviously a pleasant thing indeed, they're an endangered species. There are significantly more numbers whose square root is not even remotely close to another integer. We call these things *surds.*

So an example of a surd might be

$$\sqrt{2} = 1.414\ldots$$

We can think of expressions like $\sqrt{2}$ and $\sqrt{3}$ as *irrational* numbers, because they produce decimals that run to an infinite number of places. Even our calculators find this fact overwhelming, and they cut them off after only a dozen decimal places or so. Infinity is a difficult thing to swallow, of course. Interestingly, ancient Greek mathematicians were so suspicious of this notion that it was not allowed to be publicly mentioned. Irrational, indeed!

Thankfully, we live in a rather more enlightened era, and we very much can and do discuss surds. One frequent goal we have when working with surds is to *simplify* them. This process generally involves using a key fact, namely that:

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

We'll usually be using this fact in the reverse order: we'll take an integer, and think of it as the product of two other integers, a and b. We'll then compute their square roots individually. The process is significantly easier than it might sound. Let's take a fairly simple surd as an introductory example:

$$\sqrt{200}$$

The expression $\sqrt{200}$ is a surd because it cannot be evaluated as an integer. Predictably but a tad depressingly, entering it into your calculator won't alleviate the issue. $\sqrt{200}$ is *also* an irrational number.

We can begin simplifying this by thinking of 200 as the product of 100 and 2. Then we apply our key fact:

$$\begin{aligned}\sqrt{200} &= \sqrt{100 \times 2} \\ &= \sqrt{100} \times \sqrt{2} \\ &= 10\sqrt{2}\end{aligned}$$

Your calculator might have been useless where computing $\sqrt{200}$ was concerned, but you can certainly use it to confirm that the expressions $\sqrt{200}$ and $10\sqrt{2}$ do in fact, produce the same value. And you should also note, of course, that the answer we produced wasn't necessarily the *only* way to simplify the surd $\sqrt{200}$. We could have also used this path:

$$\begin{aligned}\sqrt{200} &= \sqrt{4 \times 50} \\ &= \sqrt{4} \times \sqrt{50} \\ &= 2\sqrt{50}\end{aligned}$$

These two results are just as valid as one another (and give the same decimal if we were to insert them into a calculator). Sometimes, the occasion calls for a different approach. In this next example, we wish to take the product of two surds and simplify it. Thankfully, the steps involved are basically the same. For instance, suppose we're given the expression

$$\sqrt{6} \times \sqrt{12}$$

and we were asked to simplify it. We would do so by producing a single surd, and then looking for factors of it that were also perfect squares. For example, in this particular instance we find that

$$\begin{aligned}\sqrt{6} \times \sqrt{12} &= \sqrt{72} \\ &= \sqrt{36} \times \sqrt{2} \\ &= 6\sqrt{2}\end{aligned}$$

Again, a calculator is useful to verifying that we've performed these steps correctly. Nonetheless, our thrilling study of surds waits for no-one and nothing. It's common to come across a number that contains both an integer and a surd. This is quite a strange sight for us, who are generally privileged to mostly encounter neat integers and so forth. We're referring to expressions along the lines of this:

$$2 + \sqrt{5}$$

We might refer to this expression as a *compound surd*, on account of its being made of two parts. It's easy to perform the basic operations - addition, subtraction, and so on - on these so-called compound surds. On the pleasant occasions where the square roots are identical, surds can be added algebraically:

$$a\sqrt{c} + b\sqrt{c} = (a + b)\sqrt{c}$$

Which makes the following expression fairly convenient to evaluate:

$$(2 + \sqrt{5}) - (4 - 2\sqrt{5})$$

Here the approach would be to subtract the integer portions of each expression, and then to subtract the surd portions of each expression. Using this method, we find that

$$\begin{aligned}(2 + \sqrt{5}) - (4 - 2\sqrt{5}) &= (2 - 5) + (1 + 2)\sqrt{5} \\ &= -2 + 3\sqrt{5}\end{aligned}$$

Multiplication of compound surds isn't overly difficult either. In fact, it works using the same principles you know from expansion of quadratic roots that are probably infinitely tedious to you by now. Therefore we find that

$$\begin{aligned}(2 + \sqrt{5})(3 - 2\sqrt{5}) &= 6 - 4\sqrt{5} + 3\sqrt{5} - (2 \times 5) \\ &= 6 - \sqrt{5} - 10 \\ &= -4 - \sqrt{5}\end{aligned}$$

So we simply follow the usual expansion process, and finish it off by collecting up the like terms. This example was especially tidy because the final term involved multiplying $\sqrt{5}$ by another $\sqrt{5}$ term, and this produced the whole number 5.

If multiplication of compound surds is mostly easy then, sadly, the same cannot be said for their division. For instance, how would we simplify *this* thing?

$$\frac{1}{1 + \sqrt{3}}$$

In fact, this type of problem requires a special trick. It's called *the difference of two squares*, and may be recalled from dreary periods spent in your drab maths classes. The difference of two squares tells us that

$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b^2$$

This can be applied to our pressing problem at hand in a neat way. What we need is to introduce a new expression. We will multiply the existing fraction by this new expression, which is:

$$\frac{1 - \sqrt{3}}{1 - \sqrt{3}}$$

We're entitled to do so because this expression is simply equal to 1. This is because its numerator and denominator are identical.

At this point, it's useful to introduce an important definition. If a surd is given by $a + \sqrt{b}$, then we'll say that the surd $a - \sqrt{b}$ is called its *conjugate*. We find that, when we use the conjugate in a division problem, something interesting happens:

$$\begin{aligned}\frac{1}{1+\sqrt{3}} &= \frac{1}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}} \\ &= \frac{1-\sqrt{3}}{1-3} \\ &= \frac{1-\sqrt{3}}{-2} \\ &= \frac{-1}{2} + \frac{\sqrt{3}}{2}\end{aligned}$$

So, by setting up a slightly different (but essentially idetical) equation using the conjugate surd of the denominator, we managed, miraculously, to produce an answer that contained no surds on the denominator. This same method goes for more complicated divisions of surds, such as

$$\frac{4+2\sqrt{2}}{1-\sqrt{5}}$$

Our first step here is always to find a conjugate surd. The conjugate we select must always belong to the surd on the denominator. So we select $1 + \sqrt{5}$ as our much-needed conjugate, and we follow the usual steps:

$$\begin{aligned}\frac{4+2\sqrt{2}}{1-\sqrt{2}} &= \frac{4+2\sqrt{2}}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}} \\ &= \frac{4+4\sqrt{2}+2\sqrt{2}+4}{1-2} \\ &= \frac{8+6\sqrt{2}}{-1} \\ &= -8-6\sqrt{2}\end{aligned}$$

And, within seconds, we reduce a nightmarish expression into a tediously common one.

Sample problems

1. Simplify the following surds. There may be several valid solutions in some cases.

(a)

$$\sqrt{125}$$

(b)

$$\sqrt{6} \times \sqrt{24}$$

2. Simplify the following expressions:

(a)

$$(-4 + 6\sqrt{7}) - (12 - 2\sqrt{7})$$

(b)

$$\sqrt{60} + \sqrt{135}$$

(Hint: you will need to simplify each surd into a common form first)

3. Simplify these expressions using the idea of the *conjugate* of a surd:

(a)

$$\frac{1}{2 - \sqrt{7}}$$

(b)

$$\frac{2 + 2\sqrt{2}}{1 + \sqrt{2}}$$

Solutions

1. (a) We observe that

$$\begin{aligned} \sqrt{125} &= \sqrt{25 \times 5} \\ &= \sqrt{25} \times \sqrt{5} \\ &= 5\sqrt{5} \end{aligned}$$

(b) We form a single surd, and then simplify as per usual:

$$\begin{aligned} \sqrt{6} \times \sqrt{24} &= \sqrt{144} \\ &= \sqrt{36 \times 4} \\ &= 6\sqrt{4} \end{aligned}$$

2. (a) The method here is to collect the "like" terms, which are the integers and common surds, so:

$$\begin{aligned} (-4 + 6\sqrt{7}) - (12 - 2\sqrt{7}) &= (-4 - 12) + (6 + 2)\sqrt{7} \\ &= -16 + 8\sqrt{7} \end{aligned}$$

(b) We solve this problem by observing that

$$60 = 4 \times 15$$

and, similarly, that

$$135 = 9 \times 15$$

hence we can find common surds easily:

$$\begin{aligned}\sqrt{60} + \sqrt{135} &= \sqrt{4 \times 15} + \sqrt{9 \times 2} \\ &= 2\sqrt{15} + 3\sqrt{15} \\ &= 5\sqrt{15}\end{aligned}$$

3. (a) We will need to apply the idea of a surd "conjugate". Here, the conjugate of the denominator is $2 + \sqrt{7}$. So we compute as usual:

$$\begin{aligned}\frac{1}{2-\sqrt{7}} &= \frac{1}{2-\sqrt{7}} \times \frac{2+\sqrt{7}}{2+\sqrt{7}} \\ &= \frac{2+\sqrt{7}}{4-7} \\ &= -\frac{2}{3} - \frac{\sqrt{7}}{3}\end{aligned}$$

(b) Once again, we need to use the conjugate of the denominator. Here, that's $1 - \sqrt{2}$. So we find that

$$\begin{aligned}\frac{2+2\sqrt{2}}{1+\sqrt{2}} &= \frac{2+2\sqrt{2}}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} \\ &= \frac{2-4}{1-2} \\ &= \frac{-2}{-1} \\ &= 2\end{aligned}$$

which is a surprisingly pleasant result.

Introduction i and complex numbers

Suppose we are leafing through a maths textbook, and we encounter a quadratic equation. This is a plausible scenario, though not necessarily a pleasant one. All quadratic equations are essentially just polynomials of degree 2. In one way or another, they all take on this generic form:

$$p(x) = ax^2 + bx + c$$

for a, b, $c \in \mathbb{R}$, and $a \neq 0$. Which really just means that, so long as a is a real, non-zero number, the equation $p(x)$ will represent a quadratic. The *solutions* of a quadratic equation are the values of x such that $p(x) = 0$. We find solutions to a quadratic equation by factorising, completing the square, or using the *quadratic formula.* You'll fondly recall that these methods were the stuff of your Level One and Two mathematics exams. In particular, the quadratic formula, is given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression inside the square root of the quadratic formula, $b^2 - 4ac$, is known as the *discriminant* of the equation, and it has a delta symbol (Δ). The we can easily compute the discriminant when we know the values of coefficients a, b and c. For example, in the equation

$$p(x) = x^2 - x - 2$$

we can quickly see that $a = 1$, $b = -1$ and $c = -2$, and so the discriminant is given by

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(1)(-2) \\ &= 9 \end{aligned}$$

Mathematicians (and, we suppose, examiners) care about the discriminant because it gives us important information about the nature of the solutions of a quadratic. Because, in this particular instance, Δ was took on a positive value, we know that this equation has *two real solutions.* In other words, there are two placed where the parabola it represents will cross the x-axis. We can also observe this by drawing $p(x) = x^2 - x - 2$, which is not taxing:

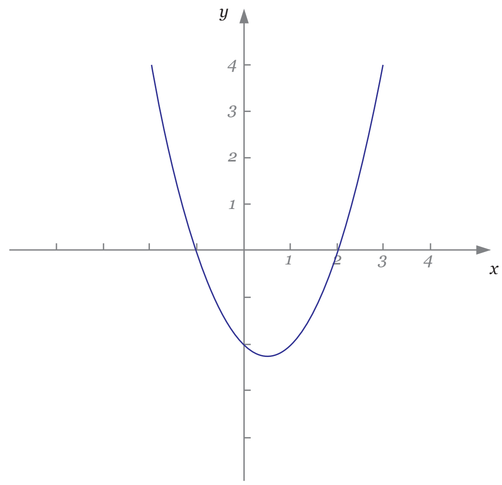


Figure 1: A graph of $y = x^2 - x - 2$

The discriminant, $b^2 - 4ac$, may take on three different kinds of values: namely, it may be positive ($\Delta > 0$), zero ($\Delta = 0$) or negative ($\Delta < 0$). The types of solutions to the quadratic equation $p(x)$ will depend on these values of the discriminant:

- If $\Delta > 0$ then $p(x)$ has *two real solutions*
- If $\Delta = 0$ then $p(x)$ has *one real solution*
- If $\Delta < 0$ then $p(x)$ has *no real solutions*

The first two cases give us plausible numbers as our solutions, even if they're sometimes a little messy. The third case, however, is more difficult to work with. This represents all those quadratic equations whose parabolas don't actually make contact with the x-axis *at all*. It's not difficult for us to imagine a parabola where this is the case. A classic example would be the equation

$$p(x) = x^2 + 1$$

whose graph is drawn below.

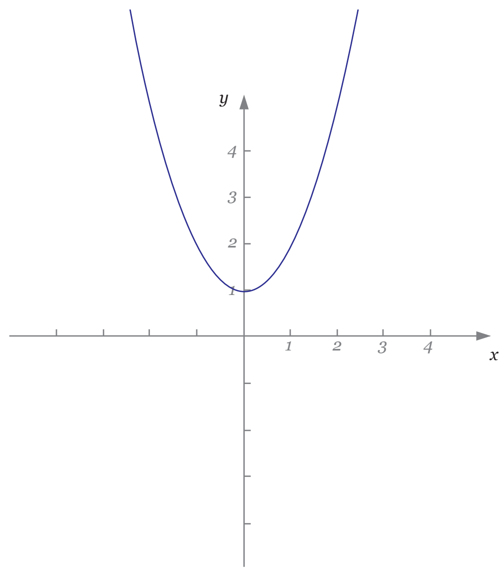


Figure 2: The graph of $y = x^2 + 1$

Clearly, this parabola does not have any x-intercepts to speak of. But, of course, sketches are often unreliable, not to mention poorly drawn. No matter, though, since we can confirm this idea by examining the discriminant value, given that $a = 1$, $b = 0$ and $c = 1$:

$$\begin{aligned} b^2 - 4ac &= (1)^2 - 4(1)(1) \\ &= -3 \end{aligned}$$

Which confirms that there are *no real solutions* that belong to this equation. Things are looking rather bleak, then.

It's frustrating that a simple equation such as this one seems to be unsolvable. Mathematicians of 16th century felt similarly. They resolved the issue in a creative way, as we shall see. Ordinarily, we solve quadratics by setting $y = 0$. So here we would observe that

$$x^2 + 1 = 0$$

which implies that we need an x such that

$$x^2 = -1$$

A *real* number, positive or negative, will always give another positive, real number when it's squared. So in order to solve this equation we need to define a new *imaginary* number, which we call i. This number is defined such that

$$i^2 = -1 \Rightarrow i = \pm\sqrt{-1}$$

Now that we have this new imaginary unit, the equation that was so vexing to us before becomes miraculously, easily solvable. We can see that if

$$x^2 = -1$$

then the solutions to the equation will be given by

$$\begin{aligned} x &= \sqrt{-1} \\ &= \pm i \end{aligned}$$

It's crucial to remember that a *plus or minus* sign must always be attached to a square root, even for instances of these imaginary numbers. Now, we should be able to factorise the original quadratic easily!

$$p(x) = x^2 + 1 \Rightarrow p(x) = (x + i)(x - i)$$

We can use regular techniques for expansion of brackets to prove that these two expressions are equivalent. For instance, we see that

$$\begin{aligned} (x + i)(x - i) &= x^2 - ix + ix - i^2 \\ &= x^2 - i^2 \\ &= x^2 - (-1) \\ &= x^2 + 1 \end{aligned}$$

and so we've shown that we must be correct. This, you may note, is some pretty heady stuff. In a few short paragraphs we've essentially made it possible to solve and entire category of previously-unsolvable equations. While it's easy for us now to appreciate the usefulness of the imaginary unit i, the mathematical community saw them as little more than amusing playthings for several centuries. Today, to a perhaps distressing extent, fields like electrical engineering rely enormously on imaginary numbers.

It eventually turns out that, whenever the number beneath a square root is a *negative*, it becomes useful to simplify the expression using i notation. For instance, suppose we're given the expression

$$\sqrt{-7}$$

and asked to simplify it. This really works similarly to the simplification surds. We need to first observe first that

$$\sqrt{-7} = \sqrt{-1 \times 7}$$

After this crucial step things become simpler. Now we simply recall that $\sqrt{-1} = i$, and we can see that

$$\begin{aligned}\sqrt{-1 \times 7} &= \sqrt{-1} \times \sqrt{7} \\ &= i\sqrt{7}\end{aligned}$$

Any number that is directly attached to an i is said to be imaginary. Real numbers and imaginary numbers are frequently found quite close to one another. We often find, for instance, that the solutions to quadratic equations contain both a real and an imaginary component. Suppose that we gave you the following, average-looking quadratic:

$$x^2 + 2x + 5 = 0$$

and demanded you solve it. It turns out that, for this equation, $\Delta = -16$, and hence the solutions *must* contain imaginary numbers. Is that all that they'll contain, though? We find the solutions, except in the simplest instances, using the quadratic formula. Let's apply it now:

$$\begin{aligned}\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-2 \pm \sqrt{-16}}{4} \\ &= \frac{-2 \pm 4i}{4} \\ &= -\frac{1}{2} \pm i\end{aligned}$$

We can now place these solutions inside sets of brackets, which will allow us to produce the following factorised form of the equation:

$$p(x) = x^2 + 2x + 5 \Rightarrow p(x) = \left(x - \left(-\frac{1}{2} + i\right)\right)\left(x - \left(-\frac{1}{2} - i\right)\right)$$

We could easily get rid of the inner brackets if we so desired:

$$p(x) = \left(x + \frac{1}{2} - i\right)\left(x + \frac{1}{2} + i\right)$$

So here, as promised, we found that the solution to the equation required both a real and imaginary element. We call these numbers *complex numbers*, which, as you no doubt can guess, will be discussed endlessly for most of the rest of the guide.

Sample problems

1. Express the following in terms of i:

   (a)

$$\sqrt{-2}$$

(b)

$$2 - \sqrt{-16}$$

(c)

$$\sqrt{-27}$$

(Hint: you should end up with an answer of the form $ai\sqrt{b}$)

2. Solve the following equations:

(a)

$$x^2 + 4 = 0$$

(b)

$$x^2 + x + 7 = 0$$

(c)

$$(x^4 - 81)$$

(Hint: first try and express this equation of the form $(x^2 + a)(x^2 - a)$)

Solutions

1. (a) Since

$$\sqrt{-2} = \sqrt{-1 \times -2}$$

we get a solution of $2i$.

(b) We can leave the real part of this complex number, 2, as it is. We perform the usual simplification on the imaginary part:

$$\begin{aligned}\sqrt{-16} &= \sqrt{-1 \times 16} \\ &= 4i\end{aligned}$$

And hence we see that

$$2 - \sqrt{-16} \Rightarrow 2 - 4i$$

(c) We begin by forming three factors of -27 in the following way:

$$\sqrt{-27} = \sqrt{-1 \times 9 \times 3}$$

Now notice that the first two of these factors can be simplified easily, so:

$$\begin{aligned}\sqrt{-1 \times 9 \times 3} &= \sqrt{-1} \times \sqrt{9} \times \sqrt{3} \\ &= 3i\sqrt{3}\end{aligned}$$

2. (a) We rearrange the equation and find the values of x such that

$$x^2 = -4$$

Hence we see that

$$\begin{aligned} x &= \sqrt{-4} \\ &= \sqrt{-1 \times 4} \\ &= \pm 2i \end{aligned}$$

(b) We apply the quadratic formula on this equation; first we observe that $a = 1$, $b = 1$ and $c = 7$. So, the solutions are given by

$$\begin{aligned} \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-1 \pm \sqrt{-27}}{2} \\ &= \frac{-1 \pm 3i\sqrt{3}}{2} \\ &= -\frac{1}{2} \pm \frac{3\sqrt{3}}{2} i \end{aligned}$$

(c) We need to first use the idea of the difference of two squares. So we observe that

$$x^4 - 81 = (x^2 + 9)(x^2 - 9)$$

Now we have two sets of quadratics. One of these we can *also* use the difference of two squares idea on:

$$(x^2 + 9)(x^2 - 9) = (x^2 + 9)(x + 3)(x - 3)$$

And so we've already found two solutions (which are *real*) to the equation. The remaining two solutions come from the factorisation of $x^2 + 9$, which gives us $(x + 3i)(x - 3i)$. So we finally can get all four solutions:

$$x^4 - 81 = (x + 3i)(x - 3i)(x + 3)(x - 3)$$

In other words, the four solutions are $3, -3, 3i, -3i$.

Operations of complex numbers

To this point, we're roughly acquainted with a few key facts about these complex numbers. One is that a complex number is made up of both a real and an imaginary component. A second is that a complex number will involve the imaginary unit, i, defined so that

$$i^2 = -1$$

These ideas will be developed in this section. First, we're ready to look at a more general form of a complex number. This is called the *rectangular form.* We normally use the letter z to denote a complex number. In rectangular form each z is defined so that

$$z = a + bi$$

for $a, b \in \mathbb{R}$. This fits neatly with the ideas we've developed about complex numbers' real and imaginary components.You should see that both a and b here are simply *real* numbers, and it is only the presence of i that creates an imaginary number here. By assigning real numbers to these unknowns a and b we create an actual complex number. So, for instance, a relatively simple complex number might be

$$z = 2 + 3i$$

We often wish to discern between the real and the imaginary portions of a complex number. We refer to the real portion as $Re(z)$, and the imaginary portion as $Im(z)$. So in our case, we would say that

$$Re(z) = 2 \text{ and similarly, that } Im(z) = 3$$

Another important fact is that a complex number need not have a real part, or an imaginary part. For instance, consider $z = -4i$. In this instance, we can see that

$$Re(z) = 0 \text{ and that } Im(z) = -4$$

and so we might call this z "purely imaginary", since it contained only an imaginary component. All the same, we would still refer to it as a complex number. To that end, even the imaginary unit i itself can be though of as an complex number! In that instance, when $z = i$, we would say that

$$Re(z) = 0 \text{ and that } Im(z) = 1$$

Likewise, a purely real number such as 52 is regardless classified as complex. So, as it turns out, all the numbers we've previously experienced during our lives have been complex numbers. Try not to feel to lied to, though.

We add and subtract complex numbers by adding or subtracting their real and imaginary parts separately. This is similar to addition and subtraction of surds from the previous section. For example, suppose we define two complex numbers such that $z_1 = 4 - i$ and that $z_2 = 3 + 2i$. Then how might we go about finding $z_1 + z_2$ and also $z_1 - z_2$. Well, we just need to group the real and imaginary parts together. So:

$$\begin{aligned} z_1 + z_2 &= (4 - i) + (3 + 2i) \\ &= (4 + 3) + (-1 + 2)i \\ &= 7 + i \end{aligned}$$

That wasn't a painful process, then, and nor is determining the difference of these two complex numbers:

$$\begin{aligned} z_1 - z_2 &= (4 - i) - (3 + 2i) \\ &= (4 - 3) + (-1 - 2)i \\ &= 1 - 3i \end{aligned}$$

As ever, it's all too easy to lose or gain minus signs when the arithmetic starts to get complicated. Brackets are therefore extremely useful when we're forced to deal with *complex* arithmetic.

Multiplication of complex numbers, happily, also happens to be extremely similar to the multiplication of surds we saw a few sections ago. All we need to do is remember that $i^2 = -1$, and the rest really takes care of itself. For instance, observe that

$$\begin{aligned} z_1 \times z_2 &= (4 - i)(3 + 2i) \\ &= 12 + 8i - 3i - 2i^2 \\ &= 12 + 8i - 3i + 2 \\ &= 14 + 5i \end{aligned}$$

Finally, division of complex numbers is very similar to division of surds. We've already seen how, in those instances, it was necessary to use the *conjugate* of a surd. So let's take a look at the related idea of the conjugate of a complex number. Suppose a complex number z is defined so that

$$z = a + bi$$

Then we would define the *conjugate* of z, $\overline{z}$, so that

$$\overline{z} = a - bi$$

Now let's look at a general case before we actually do an example. Suppose we're given the following division to solve:

$$\frac{a + bi}{c + di}$$

We examine the denominator, and deduce that its conjugate will be $c - di$. Then we re-express the problem like this:

$$\frac{a + bi}{c + di} \Rightarrow \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$$

These two expressions are equivalent, because the new fraction is simply equal to 1. This method is useful because it ensures that the thing we end up with has no complex numbers on the denominator, which would make it a real pain to work with. So let's look at how we solve $z_1 \div z_2$. If $z_2 = 3 + 2i$ then its conjugate is given by $\bar{z}_2 = 2 - 2i$. Then we change the problem so that

$$\frac{4-i}{3+2i} \Rightarrow \frac{4-i}{3+2i} \times \frac{3-2i}{3-2i}$$

Then we just need to go through the expansions, which are tedious but thankfully not too challenging:

$$\begin{aligned}\frac{4-i}{3+2i} \times \frac{3-2i}{3-2i} &= \frac{12-8i-3i+2i^2}{9-6i+6i-4i^2} \\ &= \frac{10-11i}{13} \\ &= \frac{10}{13} - \frac{11}{13}i\end{aligned}$$

This idea of complex division is extremely important, and will appear in several other places during the guide (and, in all likelihood, in your exams), so it's a good idea to practice it now. The complex conjugate will always make an appearance during these instances.

Here is another interesting idea about complex numbers. We originally defined the imaginary unit i such that

$$i^2 = -1$$

so this is one fundamental fact we know for sure about i. Given this, how could we determine the value of i^3, for instance? One useful rule we know is that

$$x^{a+b} = (x^a)(x^b)$$

This was taught at Level One and Two. Like other algebraic rules, it transitions naturally to complex numbers. Similarly, we see that

$$\begin{aligned}i^3 &= (i^2)(i^1) \\ &= (-1)(i) \\ &= -i\end{aligned}$$

So if computing i^3 was easy, could we do the same for i^4? The answer is yes. As we'll see, this case is particularly interesting:

$$\begin{aligned}i^4 &= (i^2)(i^2) \\ &= (-1)(-1) \\ &= 1\end{aligned}$$

Look at that. It's not hard to continue with this process, and see that a theme will quickly emerge. For instance, given the result we just saw, we can observe that i^5 will simply be equal to i once more.

In fact, when a raise the imaginary unit i to the power of some integer n, then the result will always be equal to one of four different things. Let's suppose that we can express the number n so that $n = 4a + b$, where a and b are whole numbers, and a takes on its maximum possible value. Then i^n will be equal to

- 1 if $b = 0$
- i if $b = 1$
- -1 if $b = 2$
- $-i$ if $b = 3$

We can use this rule to compute frankly hideous expressions, such as

$$i^{43}$$

This looks quite daunting to simplify. In fact it is laughably easy. We first need find a and b so that $43 = 4a + b$, recalling that we must select the highest integer value of a that we can. The best we can do in this instance is:

$$43 = 4(10) + 3$$

so we see that $a = 10$ and $b = 3$. Now we can re-express the original problem:

$$i^{43} = (i^4)^{10} i^3$$

The key observation to make at this point is that $i^4 = 1$. From here things progress swiftly:

$$(i^4)^{10} = 1^{10} = 1$$

The problem now becomes a great deal more straightforward:

$$\begin{aligned} i^{43} &= (1)^{10}(i^3) \\ &= i^3 \\ &= -i \end{aligned}$$

This causes of this *cyclical* nature of the imaginary unit i will be examined more thoroughly in Section Two shortly.

Sample problems

1. For the following complex numbers, determine $Re(z)$ and $Im(z)$:

   (a)
   $$z = 16 - 2i$$

   (b)
   $$z = 1.7$$

   (c)
   $$z = (2 - 4i) - (1 + i)$$

2. For $u = 1 + 5i$ and $v = 2 - 2i$ determine the values of:

   (a)
   $$uv$$

   (b)
   $$\overline{v}$$

   (c)
   $$\frac{u}{v}$$

3. (a) Simplify i^{22}

   (b) Make a general statement about i^n given that $n = 4a + 1$ for some a, $b \in \mathbb{Z}$

Solutions

1. (a) $Re(z) = 16$, $Im(z) = -2$

   (b) $Re(z) = 1.7$, $Im(z) = 0$

   (c) We first need to compute the addition and form a single complex number, so:

   $$\begin{aligned}(2 - 4i) - (1 + i) &= (2 - 1) + (-4 - 1)i \\ &= 1 - 5i\end{aligned}$$

   And hence we see that $Re(z) = 1$ and $Im(z) = -5$

2. (a) We use the normal method of bracket expansion, so that

   $$\begin{aligned}uv &= (1 + 5i)(2 - 2i) \\ &= 2 - 2i + 10i - 10i^2 \\ &= 2 + 8i + 10 \\ &= 12 + 8i\end{aligned}$$

(b) $\overline{v}$ denotes the *conjugate* of the complex number $v = 2 - 2i$ which will clearly be $\overline{v} = 2 + 2i$

(c) We solve this problem using the conjugate of v. So we see that

$$\frac{u}{v} = \frac{1+5i}{2-2i}$$

and that, according to the rules of complex number division, we can re-express the problem so that

$$\frac{1+5i}{2-2i} \Rightarrow \frac{1+5i}{2-2i} \times \frac{2+2i}{2+2i}$$

Now the problem is solvable:

$$\begin{aligned}\frac{1+5i}{2-2i} \times \frac{2+2i}{2+2i} &= \frac{2+2i+10i+10i^2}{4-4i^2} \\ &= \frac{-8+12i}{8} \\ &= -1 + \frac{3}{2}i\end{aligned}$$

3. (a) We observe that $22 = 4(5) + 2$, so in this case we see that $b = 2$. This tells us we can reduce i^{22} to -1.

   (b) Using the facts we saw in this section, for $n = 4a + 1$, it is always true that $i^n = i$. This is a sufficient solution to the problem!

Algebra of complex numbers in rectangular form

A good deal of High School mathematics, for better of worse, is devoted to solving algebraic equations. If we're lucky, these are (a) linear, and (b) contain only a single unknown. So here is an example of such an equation:

$$x + 2 = 3x - 4$$

Taking a cursory glance at this short piece of algebra, it's immediately obvious that our job is to *solve for x*. In this example, however, it was implied that the value of x would be some real number. In other, seedier scenarios, though, an unknown z could plausibly be equal to a *complex* number. These are the equations we investigate in this section.

It is an important fact that two complex numbers are equal *if and only if* both their real and their imaginary portions also happen to be equal. This idea is quite reasonable to digest. Another way of saying this is that if

$$a + bi = c + di$$

then it absolutely *must* also be true that $a = c$ and, similarly, that $b = d$. We can use this important idea in solving simple equations in which the unknown is a complex number. For example, suppose we know that

$$x + 2yi = 10 + i$$

The approach here is probably not obvious, and that's because we've never come across an equation of this form. The important thing is to appreciate that what we're seeing here is actually two seemingly-different complex numbers, one on the left of the brackets, and one on the right. Because they're *equal*, however, the real parts must be equivalent, and the imaginary parts must also be equivalent. This realisation helps a great deal. So we can say that

$$x = 10$$

and also that

$$2y = 1$$

Well, immediately we've solved for one of these variables, and found that $x = 10$. We now just need to solve the second equation for y, which is particularly easy:

$$2y = 1 \Rightarrow y = \frac{1}{2}$$

So that wasn't so bad after all. In fact, it wasn't overly different to a rudimentary piece of algebra. Similar questions can be slightly more challenging, though. For instance, here is a second example of a linear equation of complex numbers:

> Suppose that
>
> $$z(1 + 4i) = 5$$
>
> where z is a complex number.

There is a similar approach to most of these complex algebra problems and it is this: whenever we encounter an unknown z in an equation, we almost *always* should substitute z for $a + bi$, the rectangular form of a complex number. So we define $z = a + bi$ and re-write the problem in the new form of:

$$(a + bi)(1 + 4i) = 5$$

Now we can see that the question is actually asking us to determine the values of a and b, both real numbers. There seem to be several approaches to the problem at this point. One option might be to expand the left hand side, and then equate two complex numbers, like we did in the previous question. The most obvious method, though, is to algebraically divide both sides by the expression $1 + 4i$ so that we can isolate the unknowns. This also happens to be the easiest:

$$a + bi = \frac{5}{1 + 4i}$$

This ought to be ringing some bells for you. That's because we've actually seen questions extremely similar to this on in the past. We need to use the conjugate of $1+4i$, which is just $1-4i$, and we can then say that

$$a+bi=\frac{5}{1+4i}\times\frac{1-4i}{1-4i}$$

Now we only need to wade through the algebra, like we're used to doing:

$$\begin{aligned}\frac{5}{1+4i}\times\frac{1-4i}{1-4i}&=\frac{5-20i}{1-16i^2}\\&=\frac{5-20i}{17}\\&=\frac{5}{17}-\frac{20}{17}i\end{aligned}$$

So, we've shown that

$$a+bi=\frac{5}{17}-\frac{20}{17}i$$

and so, equating the real and imaginary parts of these two complex numbers we see that $a=\frac{5}{17}$ and that $b=-\frac{20}{17}$. This completes the solution, which actually wasn't too bad in the end.

We'll conclude this section with a problem that asks you to *show* a property of complex numbers. This is similar to a proof. The idea here is that we use the most general types of equation we can to show that a property holds for any *specific* examples that we might want to later choose. What this really means, though, is defining any complex numbers in a generic rectangular form $z=a+bi$ as per the previous problem.

The problem we'll be looking at is this:

> Given two complex numbers u and v, show that $\overline{u+v}=\overline{u}+\overline{v}$

We shouldn't have any problems starting out. We just need to define u and v in the general, rectangular form of complex numbers. This means writing them in forms that look like $u=a+bi$ and $v=c+di$. Of course, we chose these letters simply because they were convenient, but using x's and y's is allowed, too. From this, it clearly follows that $\overline{u}=a-bi$ and similarly, that $\overline{v}=c-di$. Now we're in good shape to begin a solution to the problem.

With a show question, our goal is to begin with the expression on the left hand side (LHS) and play around with it so that it eventually turns into the expression on the right hand side (RHS). It's not overly glamorous, in reality. We we begin with an expression for $\overline{u+v}$, which is given by

$$\overline{u+v}=\overline{(a+bi)+(c+di)}$$

Now we'll group the real and imaginary parts of this sum together, like we're used to doing:

$$\overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i}$$

Now we have what we can think of as a single complex number. We find its conjugate using the same method we're used to:

$$\overline{(a+c)+(b+d)i} = (a+c)-(b+d)i$$

We're getting close to the finish. Let's put this complex number back into two complex numbers, similar to the original values we had for u and v:

$$\begin{aligned}(a+c)-(b+d)i &= a+c-bi-di \\ &= (a-bi)+(c-di) \\ &= \overline{a+bi}+\overline{c+di} \\ &= \overline{u}+\overline{v} \\ &= \text{RHS}\end{aligned}$$

This problem was probably at Excellence level, but don't pay the meaningless NCEA-speak any mind. As we just saw, the approach didn't contain any particularly clever tricks, and the working really sorted itself out.

Sample problems

1. Solve the following equations for a complex number z in the form $z = x + iy$:

   (a)
   $$x + iy + 2 - i = 8 - 3i$$

   (b)
   $$z(2+i) = 4 + 3i$$

2. For two complex numbers, u and v, show that

   $$\overline{u \times v} = \overline{u} \times \overline{v}$$

Solutions

1. (a) We are essentially just rearranging the expressions so that on the left hand side we have only the expression $x + iy$. So we see that

   $$\begin{aligned}x + iy &= 8 - 3i - 2 + i \\ &= 6 - 2i\end{aligned}$$

(b) We begin by letting $z = x + iy$. The algebra now looks like this:

$$(x + iy)(2 + i) = 4 + 3i$$

We can now divide both sides by $2 + i$ in order to solve for $x + iy$ as desired, so:

$$\begin{aligned} x + iy &= \frac{4 + 3i}{2 + i} \\ &= \frac{4 + 3i}{2 + i} \times \frac{2 - i}{2 - i} \\ &= \frac{8 - 4i + 6i - 3i^2}{4 - i^2} \\ &= \frac{11 + 2i}{5} \\ &= \frac{11}{5} + \frac{2}{5}i \end{aligned}$$

So, aside from the maybe not-so-obvious first step, we were really just going through the motions of complex number division with that one!

2. This problem works in a similar way to the example from the notes. We begin by defining the two complex numbers in rectangular form, so that $u = a + bi$ and $v = c + di$. Let's first observe that $\overline{u} = a - bi$ and that $\overline{v} = c - di$ so that

$$\begin{aligned} \overline{u} \times \overline{v} &= (a - bi)(c - di) \\ &= ac - adi - bci - bd \\ &= (ac - bd) + (-ad - bc)i \end{aligned}$$

And so we can re express this as

$$\overline{u} \times \overline{v} = (ac - bd) - (ad + bc)i \tag{1}$$

Now we begin with the algebra. We see that

$$\begin{aligned} \overline{u \times v} &= \overline{(a + bi)(c + di)} \\ &= \overline{ac + adi + bci - bd} \\ &= \overline{(ac - bd) + (ad + bc)i} \end{aligned}$$

Now that this is in the form of a complex number, we can easily compute the conjugate:

$$\begin{aligned} \overline{(ac - bd) + (ad + bc)i} &= (ac - bd) - (ad + bc)i \\ &= \overline{u} \times \overline{v} \end{aligned}$$

by the expression we derived in (1). This completes the problem.

With that we conclude Section One of this Guide. The key skills we dearly hope you've mastered are:

- Manipulation of surds: reduction of square roots of larger integers, addition and multiplication of surds, conjugate surds and their use in division.
- i and complex numbers: the issue of a negative discriminant, using i to solve elementary quadratic equations, reducing negative square roots using i.
- Operations on complex numbers in rectangular form: real and imaginary components of a complex number, basic operations of complex numbers, use of the complex conjugate in division, reducing higher powers of i using its cyclical nature.
- Solving equations of complex numbers: equation real and imaginary components of complex numbers to solve equations, using the definition of a rectangular complex number to establish properties

2 Complex numbers in polar form

One of the more crucial facts from our previous discussions of complex numbers is that they contain both a real and an imaginary component. This is quite similar to how a point defined on the Cartesian plane - (2, -3) for instance - contains an x and a y component. We use this similarity to essentially "map" each complex number onto a set of axes. This is called an Argand diagram. Argand diagrams lead naturally to an alternative form of complex numbers known as polar form.

Argand diagrams

Let's begin by considering the complex number z, defined in rectangular form as

$$z = a + bi$$

It will be clear that $Re(z) = a$ and that $Im(z) = b$. Using this information, we can plot z on a set of axes, where the horizontal axis specifies $Re(z)$ and the vertical axis specifies $Im(z)$. So, we're really mapping the point $\big(Re(z), Im(z)\big)$ onto our new complex plane. Clearly, this would be identical to placing the point (a, b) on regular $x - y$ axes.

Thus, on an Argand diagram, where a and b are both positive numbers, the complex number $z = a + bi$ would appear like this:

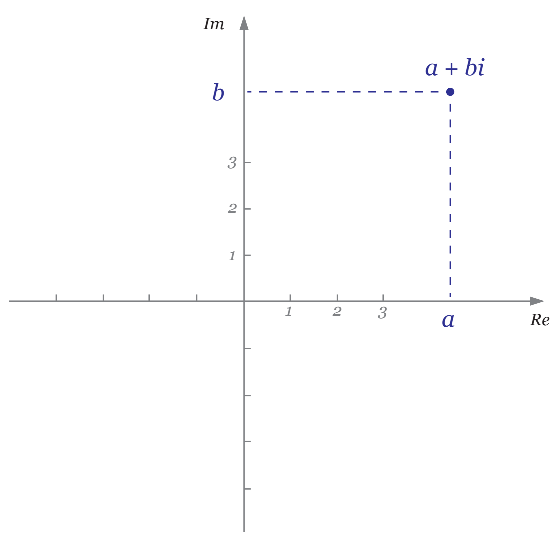


Figure 3: $z = a + bi$ plotted on an argand diagram

So there's nothing really too difficult to grasp with these Argand diagrams - they're little more, at this stage, than a tool for visualising complex numbers. Let's replace the placeholders a and b with specific values. Suppose we now define a complex number as

$$z = 2 + 3i$$

Then we can represent z on an Argand diagram like so:

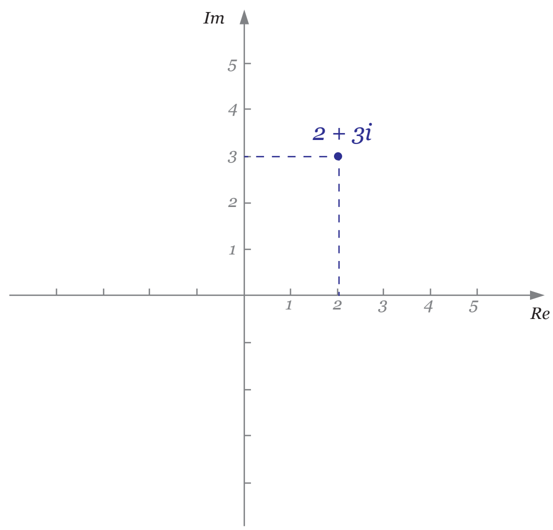


Figure 4: $z = 2 + 3i$ on an Argand diagram

Argand diagrams can also be used to add and subtract complex numbers without resorting to any actual arithmetic. Before we do this, it's a good idea to imagine that there's always an arrow running from the origin to the complex number. This is allows us to think of complex numbers as distinct *vectors*. Here, we've plotted two complex numbers, u and v, on an Argand diagram.

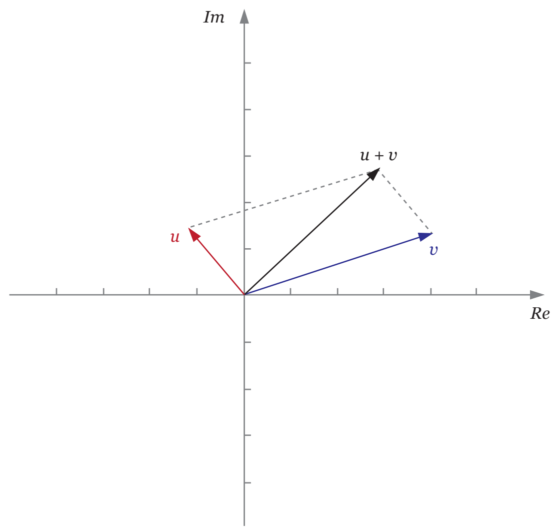


Figure 5: u, v and $u + v$ plotted in the complex plane

We've also managed to find the position of their sum, $u+v$, even without knowing the actual *values* of the these complex numbers. We did this by adding the vectors together: placing the beginning of one at the tip of another, and then drawing a final vector from the origin to this new point. You might have done this in physics classes in the past.

If we can add two complex numbers geometrically on an Argand diagram, then can we subtract them as well? Happily, the answer is yes. Here is the geometric position of $u - v$:

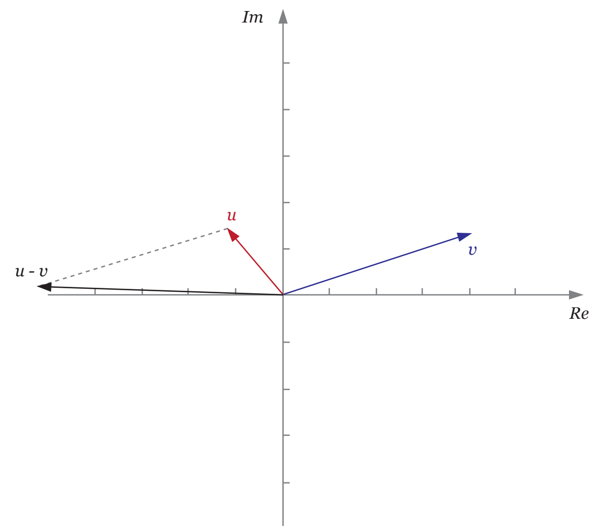


Figure 6: u, v and $u - v$, plotted in the complex plane

We find the complex number $u - v$ by thinking of the operation in the following way:

$$u - v = u + (-v)$$

Where the expression $-v$ simply represents the original v arrow with its direction reversed. So we reverse the direction of v's arrow, and add it to u in the usual manner. Also note that the position of $u - v$ will not be the same as that of $v - u$, in the same way that, for two real numbers a and b, it's not usually true that $a - b = b - a$.

One interesting thing about the complex number $u - v$, that we produces was that it lay almost horizontally. This indicates that the imaginary component of $u - v$, $Im(u - v)$, was not very large. This leads to a useful observation: when a complex number is purely real, it will lie on the real axis. Similarly, when a horizontal number is purely imaginary, it will lie on the imaginary axis.

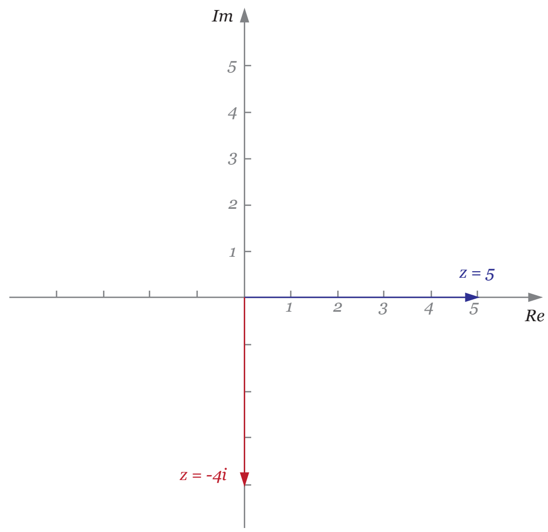


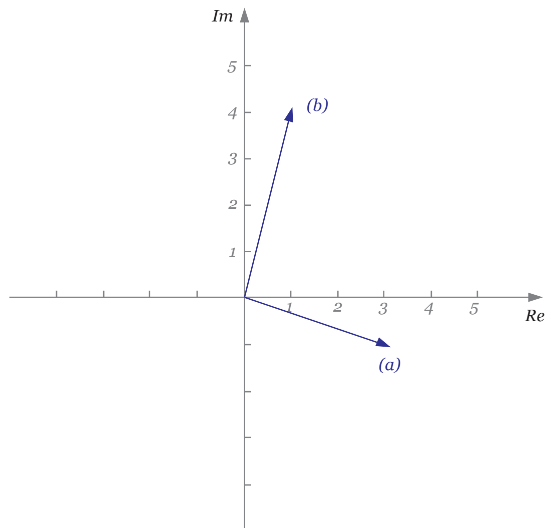
Figure 7: A purely real ($z = 5$) and purely imaginary ($z = -4i$) complex number

Sample problems

1. Plot the following complex numbers on an argand diagram:
   (a) $z = 3 - i$
   (b) $z = (2 + 3i) - (1 - i)$
2. Consider vectors $u = 1 + 2i$ and $v = -2 + 3i$. On an Argand diagram, plot the complex numbers of $u - v$ and $v - u$. What is the *angle* between these vectors? What can you deduce from this about the angle between complex numbers z and $-z$?

Solutions

1. (a) The positions for (a) and (b) are shown below:



2. The angle between these vectors is exactly 180°. We can further observe that

$$u - v = -(v - u)$$

to see that for some complex number z, the vectors of z and $-z$ will be separated by 180°. This property holds for all vectors. You may wish to check it using further examples (for instance, positive and negative real numbers).

Conversions to polar and rectangular forms

In this section, we wave a fond - but temporary -farewell to the rectangular form of complex numbers we've grown accustomed to. You'll know by now that a complex number in *rectangular form* is defined as some z such that $z = a + bi$. We've also seen that rectangular form is useful in some situations, but it also complicates things unnecessarily in other, perhaps a little like owning a car in Wellington. For instance, the expression

$$(a + bi)^{10}$$

would be lengthy and tedious to compute in rectangular form. We find the same theme for numerous other expressions of complex numbers. Thankfully, there *is* an alternative to the rectangular form, and it's called the *polar form* of a complex number. We define some z in polar form to be

$$z = r(\cos\theta + i\sin\theta)$$

where r represents the *modulus* of a complex number, $|z|$, the length of its vector, and θ represents the *argument*, $arg(z)$ the angle its vector makes with the positive part of the real axis. These baffling terms will be explained shortly.

Let's start by visualising these new ideas on the Argand diagrams of the previous section. Suppose we have an arbitrary complex number that's defined in rectangular form as

$$z = a + bi$$

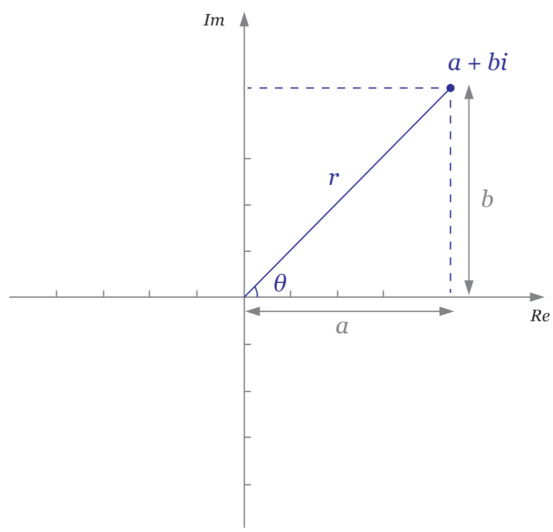


Figure 8: The polar form of $a + bi$

We can see more clearly now what the symbols r and θ geometrically represent. r, the modulus of z, is simply the length of the vector joining the origin and the position of z. So long as we know the real and imaginary components of z, we can easily find r through the Pythagorean Theorem:

$$a^2 + b^2 = r^2 \Rightarrow r = \sqrt{a^2 + b^2}$$

It is similarly easy to determine the value of θ, the argument of z. For this we use trigonometry, yet another gem of the Level One curriculum. When z lies within the first quadrant (has a positive value for a and for b) it is particularly easy to find the argument. We observe that

$$\tan(\theta) = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Generally, the argument can be specified in degrees, or in radians. In fact, in a moment of unusual generosity, NCEA will frequently permit you to use either. We will use radians for the rest of this guide, since they are generally more convenient. Let's do away with the theory and jump into an example. Our goal at this stage, remember, is simply to convert a complex number from rectangular form into polar form. We'll select the following complex number:

$$z = 2 - 3i$$

Though not entirely necessary, it's very helpful is we can visualise this complex number on an Argand diagram. This will prevent careless mistakes from occurring, especially when it comes to determining the argument of z, as we'll soon see.

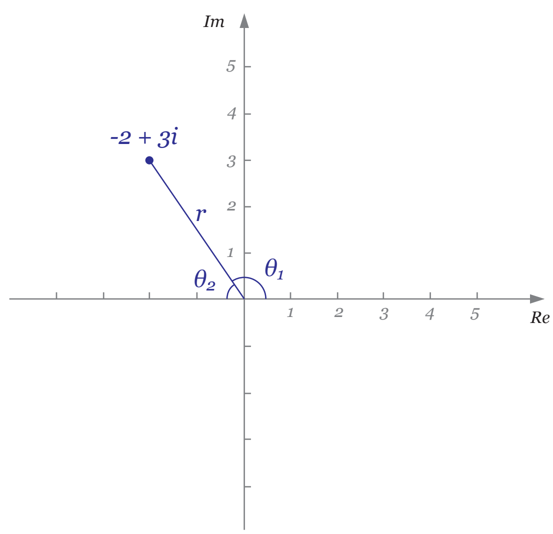


Figure 9: $z = -2 + 3i$ plotted on an Argand diagram

We'll begin by determining the value of r. The negative sign in front of the 2 will vanish during the squaring process, so we don't need to consider it here:

$$r = \sqrt{(-2)^2 + 3^2} \Rightarrow r = \sqrt{13}$$

Now we need to determine the the argument of z. Recall two important facts about $arg(z)$: the first is that this angle begins on the positive part of the real axis, and the second is that it increases in an anticlockwise manner from there. Imagine a horizontal bar that pivots about the origin and seeps around the plane, the angle θ

increasing all the way to 2π as it does.

When we apply the usual inverse tangent formula, the number we get will actually reflect the angle labelled θ_2 in the diagram. Let's find it nonetheless:

$$\theta = \tan^{-1}\left(\frac{3}{-2}\right) \Rightarrow \theta = -0.98$$

Sadly, it's fairly evident that this angle is not $arg(z)$. This presents a real problem. However, observe that z lies in the *second quadrant* of the complex plane. When this is the case we must add π radians to calculated angle in order to find the true value of $arg(z)$. Similar rules apply for the third quadrant:

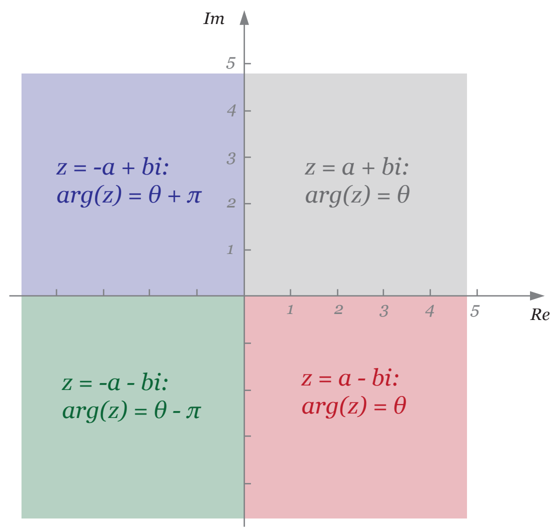


Figure 10: Rules for calculating $arg(z)$ in different quadrants

So, according to this diagram we add π to our calculated angle and find that

$$\theta = -0.98 + \pi \Rightarrow \theta = 2.2$$

Now that we've obtained correct values for r and for θ, so we can insert these into the polar formula. That formula, you will recall, is

$$z = r(\cos\theta + i\sin\theta)$$

So we find that

$$z = \sqrt{13}(\cos 2.2 + i\sin 2.2)$$

You can confirm this is correct by expanding through the brackets. Writing out complex numbers in polar form like this is quite cumbersome. Thankfully, mathematicians are an especially lazy bunch, and so we get around this tedium by compressing "$\cos\theta + i\sin\theta$" into "$\text{cis}\,(\theta)$". Hence we could also express our complex number in the form

$$z = \sqrt{13}\,\text{cis}\,(2.2)$$

So, with the exception of peculiar $arg(z)$ values, converting from rectangular to polar form is mostly straightforward. It is even easier to move in the reverse direction. For instance, suppose that

$$z = 2\,\text{cis}\left(\frac{\pi}{2}\right)$$

Recalling that this is equivalent to the expression

$$z = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

We can simply expand this expression and compute on our calculator (being careful that we are working in radians mode), so that:

$$\begin{aligned} z &= 2\left(\cos\frac{\pi}{2}\right) + 2i\left(\sin\frac{\pi}{2}\right) \\ &= 2\,(0) + 2i\,(1) \\ &= 2i \end{aligned}$$

So if $arg(z) = \frac{\pi}{2}$ then it is true that $Re(z) = 0$. In other words, this argument gives a complex number that is purely imaginary. In fact, it is true that:

$$\begin{aligned} \theta &= \frac{\pi}{2} \Rightarrow z \text{ is positive and imaginary} \\ \theta &= \pi \Rightarrow z \text{ is negative and real} \\ \theta &= \frac{3\pi}{2} \Rightarrow z \text{ is negative and imaginary} \\ \theta &= 2\pi \Rightarrow z \text{ is positive and real} \end{aligned}$$

You should confirm these statements by sketching quickly these four types of complex number on an Argand diagram and confirming the values for $arg(z)$.

Sample problems

1. Determine the polar form of the following complex numbers. Use the general formula of $z = r\,\text{cis}\,(\theta)$

   (a) $z = -1 - 4i$

   (b) $z = \sqrt{3}$

(c) $z = -2i$

2. Determine the rectangular form of the following complex numbers:

   (a) $z = 2\,\text{cis}\left(\frac{\pi}{4}\right)$

   (b) $z = 14\,\text{cis}\left(\frac{3\pi}{2}\right)$

Solutions

1. (a) Let's first determine the value of r for this complex number:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(-1)^2 + (-4)^2} \\ &= \sqrt{15} \end{aligned}$$

Now we need to find θ. Observe that z lies in the third quadrant of the complex plane. So we will need to subtract π radians from our calculated angle. This shouldn't worry you, it's a very mechanical process. So:

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{-4}{-1}\right) \\ &= 1.3 \end{aligned}$$

Hence the correct value for $arg(z)$ will be $\theta - \pi$ which is -1.8. So altogether we see that

$$z = \sqrt{15}\,\text{cis}\,(-1.8)$$

(b) This complex number is particularly quick to convert into polar form. Recall that when z is a positive, real number, then it is also true that $arg(z) = 2\pi$. All that remains is to determine the modulus of z, which is simply $\sqrt{3}$. So we see that

$$z = \sqrt{3}\,\text{cis}\,(2\pi)$$

(c) Now we have a z which is imaginary and negative. We know from earlier that, in this situation, $arg(z) = \frac{3\pi}{2}$. Similarly to the last question, we don't need to actually calculate the modulus, we simply observe that $r = 2$, and so

$$z = 2\,\text{cis}\left(\frac{3\pi}{2}\right)$$

2. (a) We see that

$$\begin{aligned} z &= 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \\ &= 2\left(\frac{1}{\sqrt{2}}\right) + 2i\left(\frac{1}{\sqrt{2}}\right) \\ &= \sqrt{2} + \sqrt{2}i \end{aligned}$$

(b) We see that

$$\begin{aligned} z &= 14\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) \\ &= 2\,(0) + 2i\,(-1) \\ &= -14i \end{aligned}$$

Operations in polar form and de Moivre's Theorem

One of the key motivations for working in polar form is that many computations that are simply exhausting in rectangular form become blissfully easy in *polar* form. Otherwise the polar form probably wouldn't be worth teaching. We began the previous section with the expression

$$(a + bi)^{10}$$

which would be nightmarish to simplify in rectangular form. However, it would be a breeze to work out if we were working in polar form. Let's think about how multiplication of complex numbers works in polar form. Consider complex numbers u and v defined as

$$u = r_1 \operatorname{cis}(\theta_1) \quad \text{and} \quad v = r_2 \operatorname{cis}(\theta_2)$$

It is a fact that their product, $u \times v$ will given by

$$u \times v = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

Hence when we multiply two complex numbers in polar form we must use the *product* of the moduli and the *sum* of their arguments. For instance, suppose we assign some values to u and v so that

$$u = 2\operatorname{cis}\left(\frac{\pi}{4}\right) \quad \text{and} \quad v = \sqrt{3}\operatorname{cis}\left(\frac{3\pi}{4}\right)$$

then we would compute $u \times v$ in the following way:

$$\begin{aligned} u \times v &= \left(2 \times \sqrt{3}\right) \operatorname{cis}\left(\frac{\pi}{4} + \frac{3\pi}{4}\right) \\ &= 2\sqrt{3}\operatorname{cis}(\pi) \end{aligned}$$

We could confirm this through an equivalent multiplication of u and v in rectangular form. Values for u, v and uv are illustrated on an Argand diagram below:

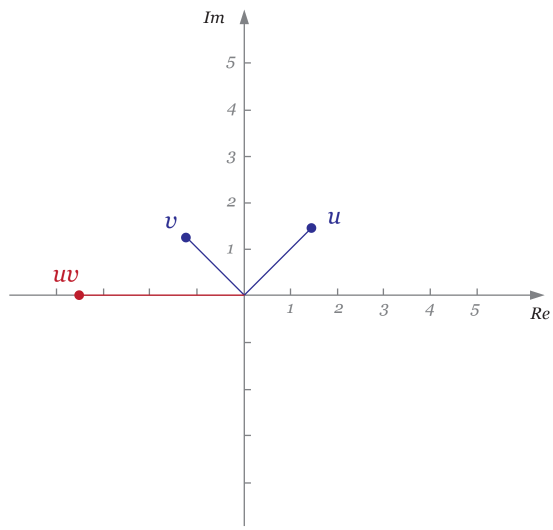


Figure 11: The product uv on a Argand diagram

So, in many cases, multiplication in polar form will create a new complex number whose argument and modulus are both larger than either of the previous complex numbers. We generally see the opposite occurring when one complex number is divided by another. Suppose again that

$$u = r_1 \operatorname{cis}(\theta_1) \quad \text{and} \quad v = r_2 \operatorname{cis}(\theta_2)$$

It is similarly true that

$$\frac{u}{v} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Let's assign the same values for u and v. Then we would now find that

$$\begin{aligned} \frac{u}{v} &= \left(\frac{2}{\sqrt{3}}\right) \operatorname{cis}\left(\frac{\pi}{4} - \frac{3\pi}{4}\right) \\ &= \frac{2}{\sqrt{3}} \operatorname{cis}\left(-\frac{\pi}{2}\right) \end{aligned}$$

which is equivalent to

$$\frac{2}{\sqrt{3}} \operatorname{cis}\left(\frac{3\pi}{2}\right)$$

of course. The process of computing a power of a complex number in polar form follows naturally from the rules of multiplication. For instance, suppose that

$$z = r\,\text{cis}\,(\theta)$$

then we find that

$$\begin{aligned} z^2 &= (r \times r)\ \text{cis}\,(\theta + \theta) \\ &= r^2\,\text{cis}\,(2\theta) \end{aligned}$$

In other words, we raise the modulus of z to the power, but *multiply* the argument of z by the power! So, in the case of $u = 2\,\text{cis}\left(\frac{\pi}{4}\right)$ we see that:

$$\begin{aligned} u^2 &= 2^2\,\text{cis}\left(2\left(\frac{\pi}{4}\right)\right) \\ &= 4\,\text{cis}\left(\frac{\pi}{2}\right) \end{aligned}$$

These ideas about powers of complex numbers can be easily generalised into a convenient formula. In this case, we're raising some complex number z to the power of n. We can observe that

$$z^n = r^n\,\text{cis}\,(n\theta)$$

The process of deriving that generalisation was relatively intuitive. We would like to find an equivalent general expression for what happens when we take the *nth root* of a complex number, $\sqrt[n]{z}$. This is more difficult, but also happens to be more enjoyable. Let's first look at the *square* root of an ordinary complex number in polar form. We'll think of one right now:

$$z = 5\,\text{cis}\left(\frac{\pi}{4}\right)$$

It's not taxing to find the square root of this complex number: after all, if we know how the process for *squaring* a complex number in polar form works, then we should be able to reverse the process. So, suppose that

$$z^2 = 5\,\text{cis}\left(\frac{\pi}{4}\right)$$

then it's similarly true that

$$r^2\,\text{cis}\,(2\theta) = 5\,\text{cis}\left(\frac{\pi}{4}\right)$$

It's clear from this statement that the only possible value of r is $r = \sqrt{5}$. But what about the possible values of θ? You may be tempted to guess that the only value of θ that will satisfy the equation

$$\cos(2\theta) = \cos\left(\frac{\pi}{4}\right) \text{ and } \sin(2\theta) = \sin\left(\frac{\pi}{4}\right)$$

will be $\theta = \frac{\pi}{8}$. In fact, there will be a second value, which will be

$$\theta = \left(\frac{\pi}{8} + \frac{2\pi}{2}\right) = \frac{9\pi}{8}$$

You can confirm this easily with a calculator set in radians mode. So it seems that there are actually *two* different square roots of z.We'll refer to them as

$$z_1 = \sqrt{5}\,\mathrm{cis}\left(\frac{\pi}{8}\right) \text{ and} z_2 = \sqrt{5}\,\mathrm{cis}\left(\frac{9\pi}{8}\right)$$

Let's examine them in the complex plane.

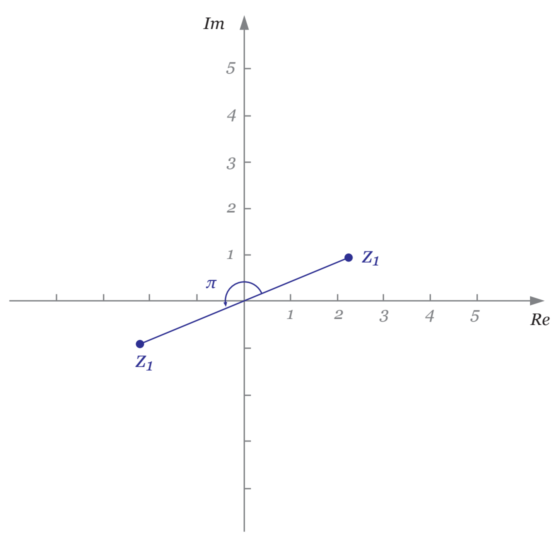


Figure 12: The two square roots of $z = 5\,\mathrm{cis}\left(\frac{\pi}{4}\right)$

These two square roots are separated by an angle of exactly π radians, or 180°. It also happens that $\frac{2\pi}{2} = \pi$ radians, or 180°. This is not a coincidence! Any time we raise some complex number to the power of n, we find that there are exactly n

roots, all spaced around the complex plane by an angle of $\frac{2\pi}{n}$ radians. This means that there will be n solutions to the problem z^n, which will be given by:

$$
\begin{aligned}
z_1 &= \sqrt[n]{r}\,\mathrm{cis}\left(\frac{\theta}{n}\right) \\
z_2 &= \sqrt[n]{r}\,\mathrm{cis}\left(\frac{\theta}{n} + \frac{2\pi}{n}\right) \\
z_3 &= \sqrt[n]{r}\,\mathrm{cis}\left(\frac{\theta}{n} + 2 \times \frac{2\pi}{n}\right) \\
& \text{all the way to} \ldots \\
z_n &= \sqrt[n]{r}\,\mathrm{cis}\left(\frac{\theta}{n} + (n-1) \times \frac{2\pi}{n}\right)
\end{aligned}
$$

This is known as the “roots of unity”, and it is a consequence of the influential *de Moivre's Theorem*. Applying de Moivre's theorem is never overly difficult. For instance, suppose that

$$z^3 = 8\,\mathrm{cis}\left(\frac{\pi}{2}\right)$$

It's immediately clear that there will be *three* solutions to this problem (i.e. three cube roots of z). They will share the same value of r, but will have different arguments spaced around the Argand diagram. We can immediately see that

$$r^3 = 8 \Rightarrow r = 2$$

Now we need to find the value of θ for each of the three cube roots. Finding the first is always easy. We can observe that

$$3\theta = \frac{\pi}{2} \Rightarrow \theta_1 = \frac{\pi}{6}$$

The remaining arguments can be found by adding $\frac{2\pi}{3}$ (the second root) and then $2 \times \frac{2\pi}{3}$ (the third root) to the first argument, so the three roots will be

$$
\begin{aligned}
z_1 &= 2\,\mathrm{cis}\left(\frac{\pi}{6}\right) \\
z_2 &= 2\,\mathrm{cis}\left(\frac{5\pi}{6}\right) \\
z_3 &= 2\,\mathrm{cis}\left(\frac{9\pi}{6}\right)
\end{aligned}
$$

If we drawn these three roots on an Argand diagram, predictably, they come out evenly spaced. The rules of evenly spaced roots can always be applied, though it's unlikely that in your exam you will be asked to find anything more strenuous than the fifth or sixth roots of a complex number z.

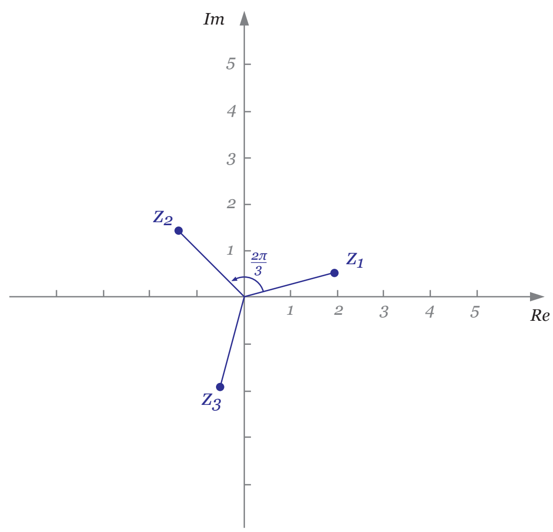


Figure 13: The three cube roots of $z = 8\,\text{cis}\left(\frac{\pi}{2}\right)$

Sample problems

1. Suppose that $u = 4\text{cis}\left(\frac{\pi}{6}\right)$ and $v = \sqrt{2}\text{cis}\left(\frac{2\pi}{3}\right)$. Evaluate:

   (a)
   $$u \times v$$

   (b)
   $$\frac{v}{u}$$

   (c)
   $$u^3$$

2. Suppose we know that for some complex number z and some real number n, it's true that $z^3 = n$. Determine the values of z using de Moivre's Theorem.

3. Using polar co-ordinates and de Moivre's theorem, evaluate

$$\sqrt{i}$$

Then, convert your solutions into rectangular form. Hint: think of i as a complex number and try evaluating it in polar form.

Solutions

1. (a) We recall that, when multiplying two complex numbers in polar form, it is necessary to take the *product* of the moduli and the *sum* of the arguments. Hence

$$\begin{aligned} u \times v &= \left(4 \times \sqrt{2}\right) \operatorname{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) \\ &= 4\sqrt{2}\text{cis}\left(\frac{5\pi}{6}\right) \end{aligned}$$

   (b) This is another straightforward operation in polar form. We now find the quotient of the moduli and the difference of the arguments. Hence:

$$\begin{aligned} \frac{u}{v} &= \left(\frac{4}{\sqrt{2}}\right) \operatorname{cis}\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) \\ &= \frac{4}{\sqrt{2}} \operatorname{cis}\left(-\frac{\pi}{2}\right) \end{aligned}$$

   The expression $\frac{4}{\sqrt{2}}$ may be simplified as $3^{\frac{3}{2}}$, but it's not really necessary.

   (c) Now we are raising the argument the power of 3, and multiplying the argument by 3. Once again, there's nothing overly difficult to this:

$$\begin{aligned} u^3 &= (4)^3 \operatorname{cis}\left(3 \times \frac{\pi}{6}\right) \\ &= 64 \operatorname{cis}\left(\frac{\pi}{2}\right) \end{aligned}$$

2. Our first task here is to represent z^3, which is itself simply another complex number, in polar form. As ever, we find an Argand diagram enormously helpful:

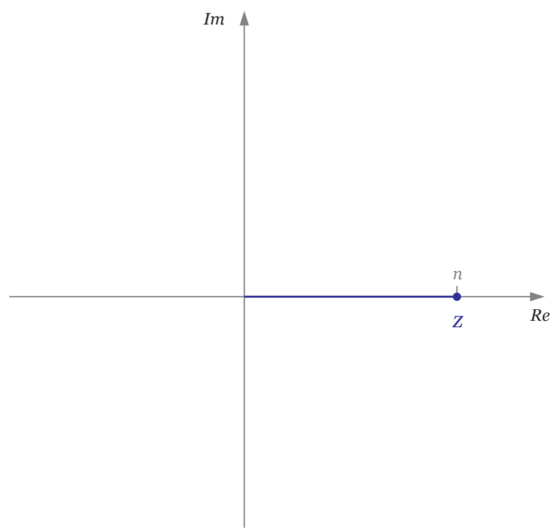


Figure 14: z^3 represented in the complex plane

So, since it is *purely real*, we would find that

$$r = \left| z^3 \right| = \sqrt{n^2} = n$$

and, similarly, that

$$\theta = arg(z^3) = \tan^{-1}\left(\frac{0}{1}\right) = 2\pi$$

Hence, as a complex number in polar form, we can define z as:

$$z = n \operatorname{cis}(2\pi)$$

Now that we've expressed z^3 in polar form, we can determine its roots using de Moivre's Theorem. There will be roots of z^3. The first will be given by

$$z_1 = \left(\sqrt[3]{n}\right) \operatorname{cis}\left(\frac{2\pi}{3}\right)$$

The two subsequent roots will both be separated by a distance of $\frac{2\pi}{3}$ radians in the complex plane. So we find that

$$z_2 = \left(\sqrt[3]{n}\right) \operatorname{cis}\left(\frac{4\pi}{3}\right)$$

and

$$z_3 = \left(\sqrt[3]{n}\right) \operatorname{cis}\left(\frac{6\pi}{3}\right)$$

the argument of z_3 is, of course, equivalent to 2π, so we may prefer to express it in that form.

3. This problem seems slightly more difficult, but so long as we approach it sensibly, we shouldn't find it too problematic. We want to begin by converting i into polar form. That's not difficult. We see that

$$r = |i| = 1$$

and that

$$\theta = \frac{\pi}{2}$$

You may wish to confirm these by representing $z = i$ on an Argand diagram. Now we observe that

$$z^2 = 1 \operatorname{cis}\left(\frac{\pi}{2}\right)$$

So, we're looking for the two square roots of our z. We apply de Moivre's theorem in the usual way. The two roots will, of course, be separated by an angle of π radians. Hence:

$$z_1 = \operatorname{cis}\left(\frac{\pi}{4}\right)$$

and

$$z_2 = \operatorname{cis}\left(\frac{5\pi}{4}\right)$$

Computing these roots in rectangular form, we see that for z_1:

$$\begin{aligned} z_1 &= \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \end{aligned}$$

and for z_2:

$$\begin{aligned} z_2 &= \cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) \\ &= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \end{aligned}$$

These are very interesting results! They can also be found working in rectangular form, which is an interesting exercise for you to try.

Algebra of complex numbers in polar form

This section deals with the miscellaneous, mostly higher-level problems about polar form. As usually happens, there's no clear theme or way to predict exactly what you might be asked during an exam. As we'll see, interestingly enough, these problems don't actually often require us to *work* in polar form! Instead, they tend just to borrow the polar ideas of the *modulus* and *argument* of a complex number. We'll begin by examining the properties of the *argument* of a complex number.

A number of important facts about the argument, θ, will already be apparent to you (and can all be confirmed by using an Argand diagram), for instance, we know that when $arg(z) = \frac{\pi}{2}$, z must be purely real, and also positive. Similarly, when $arg(z) = \pi$, we see that z is purely imaginary, and negative.

What about other values of θ, though? One important, helpful idea is that:

$$\arg(z) = \frac{\pi}{2} \Rightarrow Re(z) = Im(z)$$

We can easily prove this. Let's consider some arbitrary complex number $z = a + bi$ (notice we're working in rectangular form again). Now suppose the real and complex components are equal, so that $z = a + ai$. We can find $arg(z)$ in the traditional way, using trigonometry:

$$\begin{aligned} arg(z) &= \tan^{-1}\left(\frac{a}{a}\right) \\ &= \tan^{-1}(1) \\ &= \frac{\pi}{4} \end{aligned}$$

There we have it. Some examples of such complex numbers might be $z = 2 + 2i$, or $z = -3 - 3i$. This can all be confirmed if so so desire, using trig once more. But for now, let's move on to some applications of this powerful fact. Suppose we have the following problem:

> Two complex numbers of the form $u = 1 + 3i$ and $v = -2 + xi$ have the property that
>
> $$arg\,(u \times v) = \frac{\pi}{4}$$
>
> Determine the value of x.

You might expect that this question would require that we work in polar form for some of the time, but in fact it's not as complicated as that: the fact that the $arg(u \times v) = \frac{\pi}{4}$ is simply a kind of code for the fact that

$$Re(uv) = Im(uv) \tag{2}$$

So we can consider the complex product uv as a single complex number, like we would for an ordinary z. This fact will use useful in a moment. For now, let's try and compute $u \times v$, working in rectangular form. It's been a while since we had to do this...

$$\begin{aligned} u \times v &= (1+3i)(-2+xi) \\ &= -2 + xi - 6i + 3xi^2 \\ &= -2 - 3x + xi - 6i \\ &= (-2-3x) + (x-6)i \end{aligned}$$

Okay, so we've gone through the usual, somewhat tedious steps, and found a single complex number, which is certainly good news. If you recall, it's usually a very good idea to group the terms of a complex number so that it's in the form $z = a + bi$. So here we can see that, for the complex number uv:

$$Re(uv) = -2 - 3x \text{ and } Im(uv) = x - 6$$

This is shaping up nicely. We have two linear equations, with a single unknown, x, in each of them. We recall from equation (2) that $Re(uv) = Im(uv)$, and so we see that

$$-2 - 3x = x - 6$$

This is a straightforward equation that we can easily solve for x. We see that

$$2x = 4 \Rightarrow x = 2$$

That was a spot of fun, then. Now let's move on to something else. Let's suppose that we have a complex number $z = a + bi$, and that we allow the actual values of a and b to vary so that $|z| = 4$.

You'll recall that the expression $|z|$ is also called the *modulus* of z, and it represents the length, r, of the vector that joins z and the point $(0,0)$ on the complex plane. So, we're essentially allowing our complex number to take on *whatever* values it pleases, but we're also applying the condition that r is always equal to 4. If we draw an arbitrary z with $|z| = 4$, and allow z to vary, we find that it actually traces a *circle* in the complex plane:

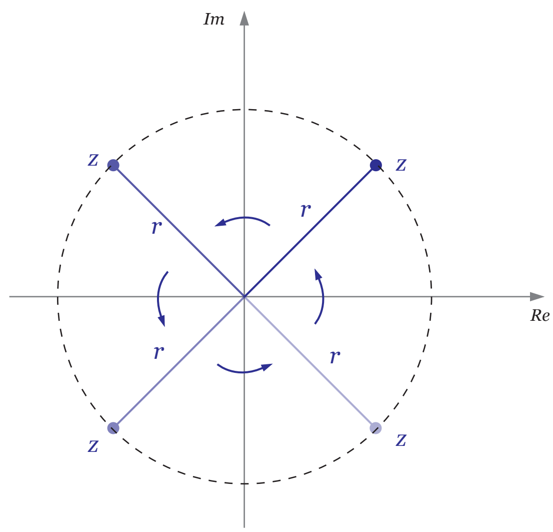


Figure 15: The locus traced out by $|z| = 4$ on an Argand diagram

What is essentially happening here is that a bar of length 4 is sweeping in all directions around the origin. It's not too hard to imagine how this might form a circle. It's also not difficult to establish mathematically of what is occurring here. We'll use rectangular form to do this. Suppose that $z = x + iy$, for an arbitrary complex number. We have imposed the condition that $|z| = 4$. We can use the Pythagorean Theorem to capture this idea as an equation.

$$|z| = \sqrt{x^2 + y^2}$$

And hence

$$\sqrt{x^2 + y^2} = 4 \Rightarrow x^2 + y^2 = 16$$

We know from our study of loci and graphs that a circle is any locus of the form

$$x^2 + y^2 = r^2$$

and so it's clear that this equation represents a circle centered at $(0, 0)$ and of radius 4. There is thus a strong relationship between the modulus of a complex number and the geometric *locus* that is traced in the complex plane. For you, this really just means another round of problems to be solved. They're never too hard, though. For instance:

Suppose we define a complex number z in rectangular form such that $z = x + iy$. Further suppose that

$$|z| = 2|z-2| \tag{3}$$

Describe fully the locus of points that this equation represents.

Despite requiring more longwinded algebraic manipulation on our part, this question is really no more difficult that the simpler one that came before it. The enormous hint comes in the first line of the question, which lets us know that we should begin by expressing z (and $|z|$ of course) in rectangular form. Let's start with what we know, which is that

$$|z| = x^2 + y^2$$

for a generically defined complex number. The way forward is, thankfully, clear from here: we simply slide this expression in for $|z|$ and $|z-2|$ in equation (3). We first observe that

$$|z| = x^2 + y^2$$

and, similarly, that

$$|z-2| = (x-2)^2 + y^2$$

Now, heading back to (3), we can form a new equation, which is

$$x^2 + y^2 = 2\left[(x-2)^2 + y^2\right]$$

Now it's a matter toughing it out, and wading through this somewhat dense algebra. We expand all the terms:

$$\begin{aligned} x^2 + y^2 &= 2\left[x^2 - 4x + 4 + y^2\right] \\ &= 2x^2 - 8x + 8 + 2y^2 \end{aligned}$$

We can see that we're heading towards the equation of a circle, or possibly an ellipse. So what we would like is to gather the x and y terms on one side, and any real numbers on the other. So, rearranging, we find that

$$x^2 - 8x + y^2 = -8$$

This equation, in one sense, *is* the correct answer. However, until it is factorised, we will be unable to properly describe the locus it forms. For that, we'll need to use the method of *completing the square* (which is always useful at any level) x's:

$$(x-4)^2 - 16 + y^2 = -8$$

Now we transfer the -16 term to the RHS:

$$(x-4)^2 + y^2 = 8$$

That's a significant improvement. We can immediately interpret this as the locus of a circle, centered at $(4, 0)$ and of radius $\sqrt{8}$. This is literally all that we would need to add to the working to get full marks in this situation.

Sample problems

1. Suppose that for some complex number $z = x + iy$ it is true that

$$arg\,(z + 3 - i) = \frac{\pi}{4}$$

   By solving for z in rectangular form, give a geometric interpretation of the locus of z.

2. Describe fully the locus of points representing z such that

$$|z + 5 - 3i| = 36$$

3. Describe fully the locus of points representing z such that

$$|z - 2|^3 = 64$$

Solutions

1. We recall that when $arg(z) = \frac{\pi}{4}$ it is similarly true that

$$Re(z) = Im(z)$$

   We will use this fact to help us move forward with this problem. We first wish to express distinctly the real and the imaginary component of $z + 3 - i$. We first consider z in rectangular form, so that:

$$z + 3 - i = x + iy + 3 - i \Rightarrow (x + 3) + (y - 1)i$$

   Now we equate the real and imaginary parts:

$$x + 3 = y - 1$$

   This is essentially the correct solution, but in order to give it a meaningful geometric interpretation, it's necessary to solve for y as we've done historically for straight lines, so we see that

$$y = x + 4$$

   and so this clearly represents a straight line with a gradient of 1 and a y-intercept of 4.

2. Now we need to go from working with the argument of a complex number to considering its *modulus*. We should be familiar with how this process might go by now.Let's first express the new complex number in rectangular form. If $z = x + iy$ then it's clear that

$$z + 5 - 3i = (x + 5) + (y - 3)i$$

Now we'll determine the modulus of this complex number, which is again straightforward:

$$\begin{aligned}|z+5-3i| &= \big|(x+5)+(y-3)i\big| \\ &= (x+5)^2+(y-3)^2\end{aligned}$$

This is shaping up well. It follows from the original problem that

$$(x+5)^2+(y-3)^2 = 36$$

and hence this clearly represents a circle of radius 6 that is centered at $(-5,3)$. This concludes the solution.

3. We've not specifically encountered this type of problem before, but it's highly solvable nonetheless. We begin, as ever, by considering the "new" complex number formed by the expression $z-2$, which is simply $(x-2)+iy$. Now we substitute this into the original equation:

$$\big|(x-2)+iy\big|^3 = 64 \Rightarrow \Big[(x-2)^2+y^2\Big]^3 = 64$$

We now take the cube root of both sides. There is only a single value of $\sqrt[3]{64}$, and it is 4. Hence

$$(x-2)^2+y^2 = 4$$

and so we see that this set of points represents another circle, which on this occasion has a radius of 4, and is centered at $(2,0)$.

With that we conclude Section Two of this Guide. The key skills we dearly hope you've mastered by now are:

- Argand diagrams: representing complex numbers as points and vectors in the complex plane, addition and subtraction of complex numbers using vector methods.
- Polar form of complex numbers: identifying the connection between polar form and the geometric properties of the Argand diagram, calculating the modulus and argument of a complex number, correcting the argument for complex numbers in the second and third quadrants of the complex plane, and conversions from polar to rectangular form.
- Operations in polar form: multiplication and division of complex numbers in polar form, raising complex numbers to powers in polar form, determining the roots of unity for complex numbers using de Moivre's theorem.
- Equations of complex numbers in polar form: using facts about the argument to solve simple problems, understanding the geometric relationship between the modulus of a complex number and the locus of a shape in the complex plane

3 Polynomial equations

Equations, both linear and non-linear, have been examined, inspected, prodded, and otherwise pored over for most of your recent education in mathematics. You perhaps never wish to regard another dubious quadratic for some time. In that particular instance, we're forced to present some unfortunate news. At the same time, however, we'll see a number of elegant theorems that make working with quadratics and cubics significantly less stressful.

Roots of quadratic equations

Polynomials are common branches of equations which include the equations for straight lines, parabolas, cubics, and a host more. A polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a_0$$

in which $a_n \dots a_0 \in \mathbb{R}$, which is simply another way of expressing that the coefficients of the x terms are some *real numbers*. We say that the *degree* of a polynomial is the highest power of x. So, in this general form here, the polynomial is of degree n. Two of the more common kinds of polynomials we come across in this standard are

- *quadratics*, polynomials of degree 2, which are of the form

$$f(x) = ax^2 + bx + c$$

- *cubics*, polynomials of degree 3, which are of the form

$$f(x) = ax^3 + bx^2 + cx + d$$

Let's first spend a small amount of time with quadratics, which are manageable to work with then cubics for obvious reasons. It's an elementary fact that all quadratics can be expressed in a *factorised* form, which entails two sets of brackets that, when expanded, will generate the polynomial for us. Sadly, as you'll all know by now, some quadratics factorise infinitely more easily than others. For instance, suppose $f(x) = x^2 + 5x + 6$. It's not difficult to work out that

$$x^2 + 3x + 6 \Leftrightarrow (x+2)(x+3)$$

where $(x+2)$ and $(x+3)$ are "factors" of the quadratic equation. Factorising is of much more use to us than simply giving an alternative form of an equation: when we factor a quadratic, we immediately see its *roots*. These roots are another way of saying the x-intercepts of the same parabola. Here, for instance, it is clear that the equation

$$x^2 + 5x + 6$$

has roots of $x = -2$ and $x = -3$. This is highly useful information. Let's now suppose that we know that the roots of a quadratic equation are two real numbers a and b. In other words, we are able to express the quadratic in the factored form

$$f(x) = (x - a)(x - b)$$

However, we may not immediately be certain what the *expanded* form of the quadratic will *be*. So, what we would like is some direct relationship between the roots of this quadratic - the real numbers a and b - and the coefficients of its expanded form. We can determine this relationship through a straightforward expansion of brackets. We start out with the factorised form of $f(x)$ that you've already seen:

$$\begin{aligned} f(x) &= (x - a)(x - b) \\ &= x^2 - ax - bx + ab \end{aligned}$$

Then, collecting terms, we find that

$$(x - a)(x - b) = x^2 - (a + b)x + ab \tag{4}$$

Recall that the real numbers a and b were the *roots* of that quadratic equation. So we can see that the coefficient of the first term, the x^2, won't depend on the value of a and b. The coefficient of the x term, however, will depend on the *sum* of the roots. Then, the coefficient of the final term without any x's will be depend on the *product* of the roots.

These relationship can be better expressed in words. When the roots of a quadratic are a and b then we observe that

$$f(x) = x^2 - (\text{sum of roots})x + (\text{product of roots})$$

Pay particular attention to the negative sign in the middle term. You may have thought that all this was essentially common knowledge, and on some level it *is*. All the same, however, this simple little formula can prove quite useful to us. For instance, suppose we know that a quadratic $f(x)$ has roots of $2 + \sqrt{3}$ and $2 - \sqrt{3}$. In other words, $f(x)$ can be expressed in the form

$$f(x) = \left(x - (2 + \sqrt{3})\right)\left(x - (2 - \sqrt{3})\right)$$

Now suppose we're asked, as we inevitably will be, to find the *expanded* form of the equation. We can clearly see that the most obvious method- to expand the terms by hand - would require a fair amount of utterly tedious arithmetic. Naturally, we wish to minimise our own workload. So, an alternative method would be to think of $2 + \sqrt{3}$ as a and $2 - \sqrt{3}$ as b from our earlier equation (4). We can easily work out the sum and product of these roots:

$$\text{Sum of the roots} = 2 + \sqrt{3} + 2 - \sqrt{3} = 4$$

$$\text{Product of the roots} = (2+\sqrt{3})(2-\sqrt{3}) = 4-3 = 1$$

The astute readers among you will notice that these calculations computed particularly easily because those roots were, in fact, conjugate surds. So now, remembering that

$$f(x) = x^2 - (\text{sum of roots})x + (\text{product of roots})$$

we can write that:

$$f(x) = x^2 - 4x + 1$$

You might wish to confirm this through the quadratic formula, or completing the square (naturally, this quadratic won't factorise pleasantly). So the power of equation (4) is clearly not to be laughed at. Let's look at yet another of its uses. Consider the equation

$$f(x) = x^2 + 7x - 16$$

We wish to find the quadratic equation whose roots are *twice* those of our $f(x)$ here. How might we approach this? One method might be to use the quadratic formula. However, this process would be tiresome, and would doubtless involve messy values. On the other hand, if we consider the *sum* and *product* of the roots, the question become laughably easy.

First, let's assign symbols a and b, as we've grown accustomed to, to represent the *roots* of $f(x)$. Keep in mind that we aren't actually interested in what those numbers *are!* Now, recalling that

$$f(x) = x^2 - (\text{sum of roots})x + (\text{product of roots})$$

we can say that

$$a + b = -7$$

and, similarly, that

$$ab = -16$$

How does this help us? We have been asked to find a new quadratic, which we'll call $g(x)$, whose roots are *twice* those of $f(x)$. So, we can think of the new quadratic's roots as being $2a$ and $2b$. In other words

$$g(x) = x^2 - (2a + 2b) + 4ab$$

We can now observe that

$$\begin{aligned} 2a + 2b &= 2(a+b) \\ &= 2(-7) \\ &= -14 \end{aligned}$$

Similarly, we can easily determine the product of $g(x)$'s roots:

$$\begin{aligned}(2a) \times (2b) &= 4ab \\ &= 4(-16) \\ &= -64\end{aligned}$$

Now we can substitute this sum and product into our equation for $g(x)$ and find that

$$\begin{aligned}g(x) &= x^2 - (-14)x - 64 \\ &= x^2 + 14x - 64\end{aligned}$$

The final thing we want to look at in this section is the *discriminant* of a quadratic equation. We have actually already encountered discriminants back in the first section on complex numbers, so we're familiar with it's main uses at this point. Now we'll be looking at using the discriminant to solve for unknowns. It's a fact that in an equation of the form

$$f(x) = ax^2 + bx + c$$

the *discriminant* of the equation is given by the expression

$$\Delta = b^2 - 4ac$$

The value of Δ determines the so-called *nature of the roots* a quadratic. We observe that:

- If $\Delta > 0$ then $p(x)$ has *two real solutions*
- If $\Delta = 0$ then $p(x)$ has *one real solution*
- If $\Delta < 0$ then $p(x)$ has *no real solutions*

We can use these facts to easily solve for unknowns in quadratics, so long as we are told the nature of the roots. For instance, consider the following problem:

> Determine the value of p if the equation
>
> $$f(x) = 2x^2 + px + p$$
>
> has no real solutions.

The key idea here is that a quadratic with no real solutions has complex roots, and a discriminant that is negative. We observe that, in this equation, $a = 2$, and $b = c = p$. Now we can use substitute these into the expression for the discriminant:

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= p^2 - 4(2)(p) \\ &= p^2 - 8p\end{aligned}$$

Now we observe that, for an equation of no real solutions, it must be true that

$$p^2 - 8p < 0$$

We solve this equation for p. This involves a simple factorisation. The *less than* sign takes the place of an *it*equals sign, and simply lingers in the middle of the equation:

$$p(p-8) < 0$$

Now we need to interpret some solutions. Because of the less than sign, the solutions won't be simple real numbers like they would be with an equals sign, but instead they'll represent some *range* of values for p. This range will clearly involve the values 0 and 8, but how do we put them together? You're perhaps groaning.

Consider the problem in this way: at this point, we have constructed the equation for a parabola, which we will call $g(x) = x^2 - 8x$ (replacing variable p with x). What we really desire, then, is the range of x values for which this parabola is beneath the x-axis. The parabola is orientated positively, and looks roughly like this:

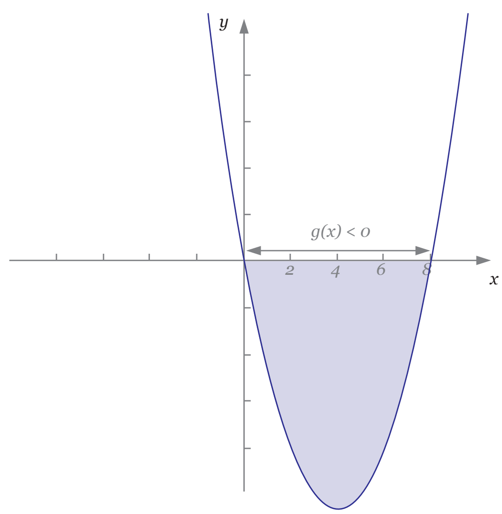


Figure 16: The range of x values for which $g(x) = x^2 - 8x$ is negative

So we can clearly see that our $g(x)$ will be negative for x values between 0 and 8. Hence the correct answer is that

$$f(x) = 2x^2 + px + p \text{ has no real solutions for } 0 < p < 8$$

So then determining the range of values did not present much cause for concern after all. As ever, we find a diagram make the problem significantly clearer.

Sample problems

1. A quadratic equation $f(x)$ has roots of $4+i\sqrt{2}$ and $4-i\sqrt{2}$. Determine $f(x)$.
2. A quadratic equation $f(x)$ is of the form
$$f(x) = x^2 - 9x + p + 7$$
in which the roots of the equation have a difference of 3, that is, $f(x)$ can be expressed in factored form as
$$f(x) = (x-a)\big(x-(a+3)\big)$$
Use this information to determine the value of p.
3. Consider the quadratic equation
$$f(x) = ax^2 + bx + c$$
Now suppose that $f(x)$ can be expressed in a factored form as
$$f(x) = a(x-\alpha)(a-\beta)$$
in which real numbers α and β are the *roots* of the polynomial.
   (a) Using an expansion, determine the relationship between the sum and product of roots α and β, and the coefficients a, b and c of $f(x)$.
   (b) Use this information to find the sum and product of the roots in the equation
$$f(x) = 2x^2 + 6x - 4$$

Solutions

1. We use the traditional sum and product of the roots method in order to find the expanded form of $f(x)$. We observe that the sum of the roots is given by
$$\begin{aligned}(4+i\sqrt{2}) + (4-i\sqrt{2}) &= 8 + i\sqrt{2} - i\sqrt{2} \\ &= 8\end{aligned}$$
And that the product of the roots is given similarly by
$$\begin{aligned}(4+i\sqrt{2})(4-i\sqrt{2}) &= 16 - 2i^2 \\ &= 18\end{aligned}$$
And hence we see that
$$f(x) = x^2 - 8x + 18$$

2. You might suspect that we will use the sum and product of the roots to solve this problem... and that is correct! We observe that the sum of the roots is given by

$$a + (a + 3) = 2a + 3$$

and that their product is given by

$$a(a + 3) = a^2 + 3a$$

Now we see that

$$2a + 3 = 9 \Rightarrow a = 3$$

We're in good shape to determine the value of p. Observing that

$$a^2 + 3a = p + 7$$

and using the recently discovered fact that $a = 3$, we easily find that

$$3^2 + 3(3) = p + 7 \Rightarrow p = 11$$

This concludes the solution.

3. (a) We'll begin with an expansion of the factored form of $f(x)$. This is somewhat tedious but not difficult:

$$\begin{aligned} f(x) &= a(x - \alpha)(x - \beta) \\ &= a(x^2 - \alpha x - \beta x + \alpha\beta) \\ &= a\left[x^2 - (\alpha + \beta)x + \alpha\beta\right] \\ &= ax^2 - a(\alpha + \beta)x + a\alpha\beta \end{aligned}$$

Now we have two expanded quadratic equations that we know are equal. We can therefore see that

$$ax^2 + bx + c = ax^2 - a(\alpha + \beta)x + a\alpha\beta$$

For these equations to be equal, the coefficients of x^2, x, and the final real-number term must all be equal. Let's first consider the sum of the roots. We can see that

$$-a(\alpha + \beta) = b$$

from which it clearly follows that

$$\alpha + \beta = -\frac{b}{a} \tag{5}$$

Excellent. Now we'll examine the product of the roots. We can see that

$$a\alpha\beta = c$$

and so it's clear that

$$\alpha\beta = \frac{c}{a} \tag{6}$$

We now have correct expressions for the sum and product of the roots of a general quadratic equation where $a \neq 1$.

(b) We apply the information of the previous question, observing that for this $f(x)$, $a = 2$, $b = 6$ and $c = -4$. So, from (5) we observe that

$$\text{Sum of roots} = -\frac{b}{a} = -\frac{6}{2} = -3$$

and, from (6), we observe that

$$\text{Product of roots} = \frac{c}{a} = \frac{-4}{2} = -2$$

Roots of cubic equations

Now that we've graduated from our re-visiting of quadratics we can graduate onto something vastly more thrilling, and that's cubics. A cubic equation is a polynomial of degree 3. That is to say, it is some equation of the form

$$f(x) = ax^3 + bx^2 + cx + d$$

for $a, b, c, d \in \mathbb{R}$ and where $a \neq 0$. Hence a cubic *must* contain an x^3 term with a real coefficient, but could in fact include as many as four or as few as one term. The simplest cubic is thus $f(x) = x^3$. Visually, cubics contain both a local maximum and local minimum value, which is impressive to behold.

One common problem associated with cubics is to determine their *roots*. In the same way that all quadratics (polynomials of degree 2) will have exactly *two* roots, it's an established and well-known fact that all cubics (polynomials of degree 3) will have exactly *three* roots. In fact, any polynomial of degree n will have exactly n roots. This is called the *Fundamental Theorem of Algebra*. Despite being admittedly quite important, all of us are testament to the fact that it's perfectly possible to learn a great deal of algebra *without* the Fundamental Theorem. The brilliant mathematician Gauss gave the first complete proof of this theorem for his Ph.D, in the precocious manner he was often associated with.

In the simplest of cases, we will be given one of the factors of a cubic equation and asked to determine the remaining two. A useful method here is known as *comparing coefficients*, and it works on the principle that two polynomials are equal *if and only if* the coefficients of their corresponding terms are equal. For instance, if it's true that

$$ax^3 + bx^2 + cx + d = 2x^3 - 4x^2 - 1$$

then it must similarly be the case that $a = 2$, $b = -4$, $c = 0$ and that $d = -1$. Now let's look at a straightforward example. Suppose we know that the cubic equation

$$f(x) = 2x^3 - 3x^2 - 17x - 12$$

has $(x + 1)$ as one of its factors. We wish to determine the remaining roots of this cubic, however, this can only be done by expressing it in fully factored form. So, how do we solve this problem?

One good tactic is to consider that a cubic is essentially just the product of a *quadratic* equation, and a linear equation. We already have a linear factor of this cubic, so the remaining factor can be thought of as some quadratic $g(x)$ such that

$$g(x) = ax^2 + bx + c$$

Now we can establish that

$$2x^3 - 3x^2 - 17x - 12 = (x + 1)(ax^2 + bx + c)$$

There are various methods that we could probably use to solve for unknowns a, b and c, and they would probably all work. However, in this case, it is easiest just to expand the RHS and compare coefficients rather than employ any algebraic tricks. This part is a little onerous, but not really hard:

$$\begin{aligned}(x + 1)(ax^2 + bx + c) &= ax^3 + bx^2 + cx + ax^2 + bx + c \\ &= ax^3 + (a + b)x^2 + (b + c)x + c\end{aligned}$$

In fact, that expansion was particularly quick. Things are shaping up rather nicely. You'll observe at this point that we have two cubic equations that, crucially, are equal. So, we know that their equivalent coefficients are equal, too. Given that

$$2x^3 - 3x^2 - 17x - 12 = ax^3 + (a + b)x^2 + (b + c)x + c$$

we can generate some linear simultaneous equations, which are

$$a = 2 \quad a + b = -3 \quad b + c = -17 \quad c = -12$$

It follows that from these that $b = -5$, which can be confirmed easily. We now see that

$$2x^3 - 3x^2 - 17x - 12 = (x + 1)(2x^2 - 5x - 12)$$

This is good. In order to find the second and third roots, we need to determine the factors of $2x^2 - 3x - 12$. There are many methods that work here. One is the quadratic formula. We see that

$$2x^2 - 5x - 12 = (2x + 3)(x - 4)$$

Hence, the three roots of the cubic are the originally given $x = -1$ and the newly discovered $x = -\frac{3}{2}$ and also $x = 4$. The solution to this rather simple problem was quite lengthy. You may wonder "if there is a quadratic formula for polynomials of degree 2 then surely there is a *cubic* formula for polynomials of degree 3?" In fact, the answer to that question is yes, however the so called "cubic formula" is enormously complicated to compute by hand. Interested parties will find that its discovery, like many things in mathematics, has a less than glamorous history. So instead of the onerous cubic formula we prefer methods such as the comparing coefficients we saw here.

In the $f(x)$ equation we just encountered, all of the three roots were real numbers. This won't always happen, however. It's entirely possible that a cubic equation have a set of *complex* roots. Before we look at these cases, there are some important facts to get out of the way that will make our lives easier. First, recall that we can view a cubic as the product of a linear and a quadratic equation. In the most recent example we saw that

$$2x^3 - 3x^2 - 17x - 12 = (x+1)(2x^2 - 5x - 12)$$

We can make some useful observations.

1. The coefficient of x^3 on the LHS comes from the product of the coefficients of the x and x^2 terms on the RHS.
2. The constant term on the LHS, -12, comes from the product of the constant terms in each set of brackets on the RHS, which here were 1 and -12.

Okay, the second idea is about the complex roots of cubics. During this standard, 100 percent of the cubic equations you will encounter will be *real-valued* cubics. This is another way of expressing that in a cubic equation

$$f(x) = ax^3 + bx^2 + cx + d$$

all of the coefficients will be real numbers, and won't contain any imaginary parts. An important consequence of this is that, whenever a cubic (or, in fact, a polynomial of any degree) has *one* complex root z, the cubic will *always* have a second complex root of $\overline{z}$, the complex conjugate of the first.

You will recall that if we define a complex number as $z = a + bi$ then its complex conjugate will similarly be $\overline{z} = a - bi$. Let's take the cubic equation

$$f(x) = x^3 - 11x^2 + 43x + c$$

and now suppose that $x = 3 + 2i$ is one of the roots of $f(x)$. Our goal is to determine the remaining roots of the cubic, and hence find the value of c, given that $c \in \mathbb{R}$.

In trying to solve a problem such as this one, alarm bells should immediately go off in our minds: whenever a cubic has a complex root, then it will *have* to have its conjugate as a second root. Hence, if one of the factors of $f(x)$ is $\big(x-(3+2i)\big)$ then a second factor will be $\big(x-(3-2i)\big)$. If we can find the product of these factors (which will be a quadratic), all that will remain is some linear factor. So we observe that

$$\big(x-(3+2i)\big)\,\big(x-(3-2i)\big)=x^2-6x+13$$

which we could quickly determine using the sum and product of roots for a quadratic. Now we can quickly see that there will be a single other linear factor for the cubic. We should equate the two, so that

$$x^3-11x^2+43x+c=(x^2-6x+13)(x+a)$$

We could have also written the linear factor as $(x-a)$, it's really up to you. This is essentially just a problem of comparing coefficients now. We first want to expand the LHS:

$$\begin{aligned}(x^2-6x+13)(x+a)&=x^3+ax^2-6x^2-6ax+13x+13a\\&=x^3+(a-6)x^2+(13-6a)x+13a\end{aligned}$$

We can now easily compare coefficients and solve for a. If

$$x^3-11x^2+43x+c=x^3+(a-6)x+(13-6a)x+13a$$

then we can immediately see that

$$a-6=-11\Rightarrow a=-5$$

or otherwise that

$$13-6a=43\Rightarrow a=-5$$

So it's easy to find the value of constant c:

$$c=13a=13(-5)=-65$$

There were two parts to this problem. The first asked us to determine the value of c, which we have done. The other asked for all three roots of the cubic. We can now do this, too. We already know that two roots are $3+3i$ and $3-2i$. Our third factor was in the form $(x+a)$, and since we now know that $a=-5$, this factor can be rewritten as $(x-5)$. Hence 5 is the third root.

Sample problems

1. One of the factors of the real-valued cubic equation

$$f(x) = 2x^3 + Ax^2 - 4x + 3$$

is $(2x - 1)$. Determine the value of A and B.

2. The real-valued cubic

$$f(x) = x^3 - 2x^2 + Ax + B$$

has $i\sqrt{3}$ as one of its roots. Determine the remaining two roots and the values of A and B.

Solutions

1. This problem is familiar territory. If we know one of the linear factors of a cubic, then we can think of $f(x)$ as the product of that linear factor and a *quadratic* factor. Hence

$$2x^3 + Ax^2 - 4x + 3 = (2x - 1)(x^2 + ax + b)$$

We knew that the coefficient of the x^2 term was 1 simply because the coefficient of the x^3 term on the LHS is also 1. Now we want to expand the RHS into some cubic equation like so:

$$\begin{aligned}(2x-1)(x^2+ax+b) &= 2x^3 + 2ax^2 + 2bx - x^2 - ax - b \\ &= 2x^3 + (2a-1)x^2 + (2b-a)x - b\end{aligned}$$

Now we form an equality of the cubic equations:

$$2x^3 + Ax^2 - 4x + 3 = 2x^3 + (2a-1)x^2 + (2b-a)x - b$$

By comparing coefficients we see that

$$2a - 1 = A \quad 2b - a = -4 \quad -b = 3$$

it follows that $b = -3$, and therefore that $a = -2$ and, finally, that $A = -5$.

2. We solve this in a similar way to the previous question. We can immediately see that if $i\sqrt{3}$ is one of the roots of the cubic then it must be true that $-i\sqrt{3}$ will be a second root. We form factors $(x - i\sqrt{3})$ and $(x + i\sqrt{3})$ and find their product:

$$(x - i\sqrt{3})(x + i\sqrt{3}) = x^2 + 3$$

and hence we see that

$$x^3 - 2x^2 + Ax + B = (x^2 + 3)(x + a)$$

Now we expand the LHS:

$$(x^2 + 3)(x + a) = x^3 + ax^2 + x + 3a$$

and see that

$$x^3 - 2x^2 + Ax + B = x^3 + ax^2 + 3x + 3a$$

and hence clearly $a = -2$, and so $A = 3$ and $B = -6$

Factor and remainder theorems

Now we look at two important polynomial theorems, which are called the *factor theorem* and the *remainder theorem* respectively. Despite having different names, they're actually rather alike. Let's start by defining a quadratic polynomial, $f(x)$, as

$$f(x) = x^2 - 3x - 10$$

Which is unexciting but suits our purpose here. Our aim with this function is to determine what *remainder* is left when we divide it by various linear expressions. For instance, what remainder (if any?) is generated if we evaluate

$$\frac{x^2 - 3x - 10}{x - 1}$$

One way of determining the remainder is to use algebraic long division. This, however, is quite a time-consuming algorithm to run by hand. A much better way is to use the remainder theorem. Let's first suppose that $(x - a)$ is *not* a factor of $f(x)$. Then there will be a real-valued remainder, R, generated when we divide $f(x)$ by $(x - a)$. It happens that we can also find R by computing

$$R = f(a) = (a)^2 - 3(a) - 10$$

So let's assign a the value of 1, which means we wish to determine the remainder when we evaluate the expression

$$\frac{x^2 - 3x - 10}{x - 1}$$

Once again, according to the remainder theorem, the determining the remainder through long division is equivalent to simply computing $f(1)$, which we can easily do ourselves:

$$\begin{aligned} R &= f(1) \\ &= (1)^2 - 3(1) - 10 \\ &= -12 \end{aligned}$$

Hence we find that a the remainder from this division problem is -10. Another interpretation of this result is that $(x - 1)$ is *not* a factor of $f(x)$. So, in its most basic form, the remainder theorem is a simple but powerful piece of theory. It leads to a number is interesting problems (many of which you may be asked during an exam). For instance:

For a real valued quadratic equation $f(x) = x^2 + px + q$ it is found that:

- When $f(x)$ is divided by $(x - 1)$ there is remainder of -4

- When $f(x)$ is divided by $(x+2)$ there is a remainder of -7

Use this information to determine the values of p and q

The approach here requires that we use the remainder theorem. These two pieces of information can be interpreted directly as meaning that:

$$f(1) = 1 + p + q = -4 \tag{7}$$

and similarly that

$$f(-2) = 4 - 2p + q = -7 \tag{8}$$

We now have two equations of two unknowns that must be simultaneously solved. Thankfully this is not particularly taxing. One method would be to multiply equation (7) by 2, and then add them together. We find that

$$2 + 2p + 2q = -8$$

Now, we add this to (8), and we find that

$$2 + 2p + 2q + 4 - 2p + q = -15 \Rightarrow q = -7$$

from which it follows that $p = 2$. So the full quadratic must have been given by

$$f(x) = x^2 + 2x - 7$$

You may wish to confirm this, once again, using the remainder theorem method. One final thing to note is that the remainder theorem can easily applied to polynomials of degree three (cubics) or higher (quartics, and so on), using exactly the same methods that we have described here.

The *factor theorem* follows naturally form the ideas of the remainder theorem. We have already discussed that when a linear expression is a factor of a polynomial, the remainder R will simply be equal to zero. Thus, the factor theorem simply expresses the idea that

$$(x-a) \text{ is a factor of } f(x) \Rightarrow f(a) = 0$$

For instance:

Determine whether $(x+3)$ is a factor of $x^3 - 5x - 2$

Answering this question is very straightforward. We simply observe that

$$\begin{aligned} f(-3) &= (-3)^3 - 5(-3) - 2 \\ &= -27 + 15 - 2 \\ &= -14 \neq 0 \end{aligned}$$

And thus the obvious conclusion is that *no*, $(x-3)$ is not a factor of $f(x) = x^3-5x-2$. So using the factor theorem is no more demanding that the previous remainder theorem was. Sometimes, more interesting problems do emerge that require the factor theorem. For instance, suppose we define a cubic equation as

$$f(x) = x^3 + Ax^2 + Bx - 15$$

Now suppose we know that the quadratic expression $(x^2 - 4x - 5)$ is already a factor of $f(x)$. How can we find the values of A and B?

This is, in fact, just the kind of problem we might previously have solved using the method of comparing coefficients. That method is still entirely valid here, but it's useful if we have more than one way of approaching such a problem. Our goal is to factorise the factor $(x^2 - 4x - 5)$ so that we might apply the factor theorem on $f(x)$. We can do this quite easily:

$$x^2 - 4x - 5 \Rightarrow (x - 5)(x + 1)$$

We now can see that the linear expressions $(x - 5)$ and $(x + 1)$ must also be factor of $f(x)$. At this point we apply the factor theorem in the usual way:

$$f(5) = 125 + 25A + 5B - 15 = 0$$

and also

$$f(-1) = -1 + A - B - 15 = 0$$

It's important to recall that the cubing of a negative number results in *another* negative number. We now, once again, are left with two simultaneous equations of two variables, that can be solved via whatever method you desire. We eventually find that

$$A = -1 \quad \text{and} \quad B = -17$$

and hence the correct cubic equation is

$$f(x) = x^3 - x^2 - 17x - 15$$

Sample problems

1. Determine the remainder that is produced when the cubic equation

   $$f(x) = 6x^3 - 5x^2 - x + 14$$

   is divided by $(x + 4)$

2. Show that $(x - a)$ is a factor of

   $$f(x) = x^3 - ax^2 + ax - a^2$$

3. Suppose a cubic equation is defined to be

$$f(x) = x^3 + Ax^2 + Bx - 12$$

and that $(x - 6)$ and $(x + 1)$ is known to be a factor of $f(x)$. Use the factor theorem to determine A and B.

Solutions

1. This is straightforward. We simply wish to evaluate $f(-4)$. We see that

$$f(-4) = 6(-4)^3 - 5(-4)^2 - (-4) + 14 = -446$$

2. We simply need to show that $f(a) = 0$. So we observe that

$$\begin{aligned} f(a) &= (a)^3 - a(a)^2 + a(a) - a^2 \\ &= a^3 - a^3 + a^2 - a^2 \\ &= 0 \end{aligned}$$

This is all that is required.

3. We begin by observing that $f(6)$ and $f(-1)$ will both equal zero when evaluated. So we see that

$$f(6) = 216 + 36A + 6B - 12 = 0$$

and similarly that

$$f(-1) = -1 + A - B - 12 = 0$$

We now, once again, find ourselves with two simultaneous equations of two variables. These can easily be solved for A and B. We find that $A = -7$ and that $B = -16$.

With that we conclude Section Three of this Guide. The key skills we dearly hope you've mastered by now are:

- Roots of quadratic equations: determining the expanded equation of a quadratic based upon its factorised form, solving simple problems using the sum and product of the roots theorem, using the discriminant to solve problems of unknowns in quadratics.
- Roots of cubic equations: using the factorised form of a cubic to solve for unknowns, examining the relationship between the roots of a cubic and its expanded coefficients, using complex conjugates in cubic equations.
- Factor and reminder theorems: determining remainders of polynomials without algebraic long division, solving for unknowns using the remainder and factor theorems.